Name: **Enrolment No:**	

UNIVERSITY OF PETROLEUM AND ENERGY STUDIES

End Semester Examination, Dec 2023

Course: Advanced Algebra **Semester: V**
Program: B. Sc. (Hons.) Mathematics + Int. B.Sc.-M.Sc. Mathematics **Time: 03 hrs.**
Course Code: MATH 3031 **Max. Marks: 100**
Instructions: All questions are compulsory.

SECTION A
(5Qx4M=20Marks)

S. No.		Marks	CO
Q1	Find the cardinality of set of all such permutations in symmetric group S_6 each having order equal to 6.	**4**	**CO1**
Q2	Explain why the group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$ is not isomorphic to the group $\mathbb{Z}_{18}$; where the symbol $\oplus$ denotes the external direct product.	**4**	**CO1**
Q3	Determine the quotient group $\frac{S_3}{<(1\ \ 2)>}$; where S_3 is the group of permutations on the set $\{1,2,3\}$ and $< \beta >$ denotes the cyclic subgroup generated by β.	**4**	**CO1**
Q4	List all primes p such that the system of equations: $$5x + 3y = 4 \text{ and } 3x + 6y = 1$$ have a unique solution in the field $\mathbb{Z}_p$.	**4**	**CO2**
Q5	Consider a set G consisting of all $n \times n$ real square diagonalizable matrices over field $\mathbb{R}$. Give reasons to justify whether it forms a ring or not.	**4**	**CO2**

SECTION B
(4Qx10M= 40 Marks)

Q 6	Determine the number of elements of order 5 in $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2$ where $G_1 \oplus G_2 \oplus G_3$ denotes the external direct product of three groups G_1, G_2 and G_3.	**10**	**CO1**
Q7	Find all distinct subgroups each having order 3 of the group $G = \mathbb{Z}_9 \oplus \mathbb{Z}_3$.	**10**	**CO1**
Q8	Consider the ring $M_n(\mathbb{R})$ of all square real matrices of order n and its subset defined as: $$S = \{A \in M_n(\mathbb{R}) \mid A^T = A\}$$ Prove or disprove that S is a subring of $M_n(\mathbb{R})$.	**10**	**CO2**

Q9	Prove that 3 is not prime in the ring $\mathbb{Z}[\sqrt{-41}] = \{a + ib \mid a, b \in \mathbb{Z}\}$. **OR** Prove that 3 is irreducible in the ring $\mathbb{Z}[\sqrt{-41}] = \{a + ib \mid a, b \in \mathbb{Z}\}$.	**10**	**CO3**
SECTION-C **(2Qx20M=40 Marks)**			
Q10	Suppose $n \in \mathbb{Z}_{>0}$. Consider ring $R_{(n)}$ defined as $R_{(n)} = \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\}$ and $\alpha \in R_{(n)}$. Prove or disprove: (a) $\alpha = 3$ is irreducible in $R_{(2)}$. (b) $\alpha = 2$ is not prime in $R_{(5)}$.	**20**	**CO3**
Q11	Consider the set $S = \{p(x) \in \mathbb{Z}[x] \mid p(-1) = p(1) = 0\}$. where $\mathbb{Z}[x]$ is the ring of polynomials with the usual operations of pointwise addition and pointwise multiplication. Prove that S is an ideal of $\mathbb{Z}[x]$. Is S a prime ideal? Is S maximal in $\mathbb{Z}[x]$? Give reasons for your choice. **OR** Prove that $\frac{R}{J}$ is a PID whenever R is a PID and J is an ideal such that $ab \in J \Longrightarrow a \in J \text{ or } b \in J.$	**20**	**CO2**