# UNIVERSITY OF PETROLEUM AND ENERGY STUDIES

**End Semester Examination, December 2019**

**Programme: B.Sc. (Hons.) Mathematics** — **Semester: III**
**Course Name: Multivariate Calculus** — **Max. Marks: 100**
**Course Code: MATH 2029** — **Duration: 3 Hrs.**
**No. of page/s: 02**

**Instructions:**
Attempt all questions from **Section A** (each carrying 4 marks); all questions from **Section B** (each carrying 10 marks) and all questions from **Section C** (carrying 20 marks).

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| | **Section A**<br>**( Attempt all questions)** | | |
| **1.** | Find the first order partial derivatives of the function $F(x,y) = \int_y^x \cos(e^t)\, dt$. | **[4]** | **CO1** |
| **2.** | Calculate the iterated integral $\int_0^1 \int_1^2 \frac{xe^x}{y} dy\, dx$. | **[4]** | **CO2** |
| **3.** | Determine whether or not the vector field $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + (x-2)\,\mathbf{j}$ is conservative. | **[4]** | **CO3** |
| **4.** | Evaluate $\int_C y \sin z\, ds$, where $C$ is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$. | **[4]** | **CO3** |
| **5.** | If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $P, Q$, and $R$ have continuous second-order partial derivatives, show that div curl $\mathbf{F} = 0$. | **[4]** | **CO3** |
| | **SECTION B**<br>**(Q6, Q7, and Q8 are compulsory. Q9 has internal choice)** | | |
| **6.** | Show that the function $f(x,y) = \sqrt{\lvert xy \rvert}$ is not differentiable at the point $(0,0)$, but the first order partial derivatives exist at the origin and have the value 0. | **[10]** | **CO1** |
| **7.** | Evaluate $\iint_D xy dA$, where $D$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. | **[10]** | **CO2** |
| **8.** | Evaluate $\int_C x^4 dx + xy dy$, where $C$ is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$. | **[10]** | **CO3** |
| **9.** | Wheat production $W$ in a given year depends on the average temperature $T$ and the annual rainfall $R$. Scientists estimate that the average temperature is rising at a rate of $0.15°\mathrm{C}/\mathrm{year}$ and rainfall decreasing at a rate of $0.1\ \mathrm{cm}/\mathrm{year}$. They also estimate that, at current production levels, $\frac{\partial W}{\partial T} = -2$ and $\frac{\partial W}{\partial R} = 8$.<br>a. What is the significance of the signs of these partial derivatives?<br>b. Estimate the current rate of change of wheat production, $\frac{dW}{dt}$.<br>**OR** | **[10]** | **CO1** |

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| | Consider the problem of maximizing the function $f(x,y) = 2x + 3y$, subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.<br>a) Try using Lagrange multipliers to solve the problem.<br>b) Does $f(25,\ 0)$ give a larger value than the one in part a)?<br>c) Solve the problem by graphing the constraint equation and several level curves of $f$.<br>d) Explain why the method of Lagrange multipliers fails to solve the problem.<br>e) What is the significance of $f(9,4)$? | | |
| **SECTION C**<br>**(Q10 is compulsory. Q11A and Q11B have internal choices)** | | | |
| **10.A** | Use the Divergence Theorem to calculate the flux of **F** across $S$. $\mathbf{F}(x,y,z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$, $S$ is the surface of the tetrahedron enclosed by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where $a, b,$ and $c$ are positive numbers. | **[10]** | **CO3** |
| **10.B** | Use Stokes' Theorem to evaluate $\int_C \mathbf{F}.d\mathbf{r}$. In this case $C$ is the oriented counterclockwise as viewed from above. $\mathbf{F}(x,y,z) = \mathbf{i} + (x + yz)\,\mathbf{j} + (xy - \sqrt{z})\,\mathbf{k}$, $C$ is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant. | **[10]** | **CO3** |
| **11.A** | Use polar coordinates to find the volume of the solid enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$.<br><br>**OR**<br><br>Use the change of variables $x = u^2 - v^2, y = 2uv$ to evaluate the integral $\iint_R y\,dA$ where $R$ is the region bounded by the $x$- axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$. | **[10]** | **CO2** |
| **11.B** | If the improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely, then show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\; e^{-(x^2+y^2+z^2)} dx\,dy\,dz = 2\pi$ .<br><br>**OR**<br><br>Use cylindrical coordinates to show that the volume of the solid bounded above the sphere $r^2 + z^2 = a^2$ and below the cone $z = r\cot\phi_0$ (or $\phi = \phi_0$), where $0 < \phi_0 < \frac{\pi}{2}$, is $V = \frac{2}{3}\pi a^3(1 - \cos\phi_0)$. | **[10]** | **CO2** |