# FIXED POINT AND APPROXIMATE FIXED POINT THEOREMS WITH APPLICATIONS

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Dehradun, Uttarakhand, India Shweta Sachdeva

# DECLARATION

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text.

Date: May 30, 2014 Shweta Sachdeva

Place: Dehradun

**UNIVERSITY OF PETROLEUM & ENERGY STUDIES**

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# CERTIFICATE

This is to certify that the thesis entitled "**Fixed Point and Approximate Fixed Point Theorems with Applications**" submitted by **SHWETA SACHDEVA** to **University of Petroleum and Energy Studies**, in partial completion of the requirements for the award of the degree of Doctor of Philosophy (Science-Mathematics), is an original work carried out by her under our joint supervision and guidance.

It is certified that the work has not been submitted anywhere else for the award of any other degree or diploma of this, or any other university.

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*To Sai Baba.*

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# Preliminaries and Some Fundamental Results

## 1.1 Introduction and Literature Review

In the mid of twentieth century an independent branch of Mathematics was developed which was known as Nonlinear analysis. It was considered as combination of functional and variational analysis by a famous mathematician of that time names Andrew Browder. Its nonlinear results have wide range of application in subjects like Physics, Chemistry, Biology and also in Economics which leads to nonlinear models. The fixed point theory has been considered as important branch in nonlinear analysis and developed in the process of advancement of the same.

Mathematicians and researchers have been considerably attracted by the fixed point theory and through this theory many interesting results have been derived during this period. Banach fixed point theorem has fascinated many researchers since 1922. It is a continued active field of research in the present time as well. Let $X$ be a nonempty set and $T : X \to X$ then a point $z \in X$ is called a fixed point of $T$ if $Tz = z$.

Theorems dealing with existence and construction of a solution to operator equation $Tz = z$ form a part of fixed point theory.

The existence of a fixed point is an intrinsic property of a map, which involves many necessary and sufficient conditions in a mixture of algebraic, order theoretic or topological properties of the map or its domain.

The existence and properties of fixed points are used as very important tools to analyze the existence and uniqueness of solution of various mathematical models namely differential, integrals, partial differential equations and variational inequality etc. (see, for instance [5], [6], [9], [20], [23], [32], [52], [58], [67], [69], [70], [75], [78], [81], [82], [83], [84], [107], [148], [162], [179], [180], [183], [191], [193], [195], [198], [199],

[200] and references thereof).

Approximate or exact solutions of boundary value problems could be derived through fixed point theorems of ordered Banach spaces for details one can refer to H.Amann [4], Collatz [46], Franllin [66], Karamardian [93], Lions [110], Martin [115], Mercier [116], Peitgen and Walther [129], Robinson [156], Smart [175], Swaminathan [179], Tartar [182] and Waltman [192].

The origin of the Fixed point theory, an important branch of nonlinear functional analysis, which dated back to the latter part of the nineteenth century, rests in use of successive approximations to the existence and uniqueness of solutions, particular to differential equations. This method is linked with the names of famous mathematicians such as Cauchy [35], Lipschitz [113], Liouville [112], Peano [127, 128] and especially, Picard [131].

Picard [131] was the pioneer in using the fixed point theoretic approach in his work. Polish mathematician Stefan Banach [9, 10] is associated for taking the credit for placing the idea of fixed point theorems into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations.

Banach also discovered the fundamental role of metric completeness. It is a specific property of the metric which is shared by all the space commonly exploited in analysis. For many years, activity in metric fixed point theory was limited to minor extensions of Banach's contraction principle and its manifold applications. The pioneering work of Browder [30, 31] gave new impetus to the theory. Since 1965, a significant development had started for existence theorem of Browder [29, 32], Göhde [68], Kirk [103, 104] and the early metric results of Edelstein [60, 61]. By the end of the decade, a rich fixed point theory for non-expansive mappings was clearly emerging and it was equally clear that such mappings played a fundamental role in many aspects of nonlinear functional analysis. Investigation of fixed point theory in sense of quality and amount in metric space has greatly increased in the 1970's. The important description of development in this period proved the existence of fixed point theorems by using more generalized contractive mappings. Idea of more generalized contractive mappings were designed by Bianchini [22], Caristi [34], Sehgal [163], Chatterjea [36], Hardy and Rogers [73], Ćirić [41–44] and Guseman [72].

Generalized contractive mappings, which are the perception of weak commutativity and compatible mappings was introduced by Sessa [164, 165] and Jungck [85, 86] in 1980's. This proved to be the turning point in the fixed point's arena and brought an elevation in the study of common fixed point theorems for mappings satisfying some contractive conditions. Thereafter, a torrent of common fixed point theorems were shaped by various authors for example Sessa [165], Park and Bae [123], Fisher [65], Kang and Cho [89], Murthy [87] and Pathak [125] by using the enhanced notion of compatibility of mappings. Since 1990's, the study of fixed point theory is focused more on the existence of fixed point in metric space, in generalized metric space and $D$-metric space using contractive mappings by the then researchers namely Kada-Suzuki-Takahashi [88], Stojakovic [176, 177], Dhage [54–57], Ume [187–189] and Rhoades [142, 146, 153].

To study the more generalized contractive mapping and different application in analysis is the focused subject of the fixed point theory. Fixed point theorems gives those conditions under which mappings (single or multivalued) have solutions.

The Dutch mathematician L.E.J. Brouwer in 1912 which stated that "A continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point", which is a major classical result in fixed point theory. This result points to an excellent development of fixed point theory in normed spaces. For details, one may refer to Agarwal [2], Barnsley [11, 15, 16], Berinde [20], Bonsal [23], Cai and Paige [33], Dugundji and Granas [58], Granas and Dugundji [70], Istrătescu [81], Kirk and Sims [105], Rhoades [143–154], Rus [158], Singh [167–171], Smart [175], Yadav [196] and Zeidler [198].

In 1922, the great Polish mathematician Stefen Banach [9] proved a wonderful theorem properly known as "Banach's contraction principle (BCP)" which stated that every contraction mapping of a complete metric space to itself has a unique fixed point. Symbolically a contraction mapping $T$ given by $d(Tx, Ty) \leq k.d(x, y)$ for all $x, y \in X$ and $0 \leq k < 1$, defined on a complete metric space $(X, d)$ has a unique fixed point in $X$. Banach's contraction principle and its modified applications were limited to minor extensions for many years. Therefore, numerous generalizations of the BCP either by weakening the contraction conditions of the mappings or by extending the structure of the ambient spaces, have been obtained.

A continuous self mapping of a compact convex subset of a Banach space has at least one fixed point was stated by Schauder [161] in 1927 which extended the above result. Another approach of generalizing the BCP involves replacement of $d(x, y)$ in the condition $d(Tx, Ty) \leq k.d(x, y)$ by a combination of distances between $x$, $y$ and their images. Rhoades [146] had given a fine survey and comparison of various generalized contraction. In this survey the general fixed point theorems were either stated or proved. Further, Meszaros [117] also studied various contractive type mappings. The most general among contraction type condition for self map $T$ of a metric space $(X, d)$ is due to Ćirić [43]. One can also refer to Hegedus [74], Park and Rhoades [124], Collaco and Silva [45] and references therein for a deep understanding of the above literature. Ekeland [62] gave an interesting generalization of the Banach contraction principle.

In 1965, Chu and Diaz [39] generalized the Banach fixed point theorem by proving that if $T^n$, the $n^{th}$ iterate of a self mapping $T$ defined on a complete metric space $X$ is a contraction for some positive integer $n$. Then $T$ has a unique fixed point in $X$. The only draw back of Banach fixed point theorem despite of numerous advantages is that it requires the continuity of mapping $T$ throughout the space. In 1968, Kannan [90] established result that does not require the continuity of mapping $T$ by using the contractive condition $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$ where $0 \leq k < \frac{1}{2}$, then self-map $T$ has a unique fixed point in complete metric space $(X, d)$. Gupta and Rangnathan [71] have proved a fixed point theorem for a mapping which is not necessarily continuous.

The initiation of fixed point theory in computer science by Tarski [181] enhances its applicability in different domains. Cai and Paige [33] found that fixed points are involved in program derivation which influence dramatically the construction, reliability, maintenance and extensibility of a software.

Fixed point theorems have fascinated many generalizations in various spaces such as cone metric space, fuzzy metric space, $b$-metric spaces etc. Indeed, the investigations relate mainly to the problem of presence and locating exact fixed points of maps on various settings. The theory of approximate fixed point is more apt for actual applications and approximating a fixed point solution in numerical practice.

## 1.2 Metric Spaces

Let $X$ be a nonempty set and $d : X \times X \to R$ a mapping. Then $d$ is called a metric on $X$ if the following properties hold:

**(d1)** $d(x, y) \geq 0$ for all $x, y \in X$,

**(d2)** $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$,

**(d3)** $d(x, y) = d(y, x)$ for all $x, y \in X$,

**(d4)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric $d$ at $(x, y)$ is called a distance between $x$ and $y$, and the ordered pair $(X, d)$ is called Metric Space.

**Definition 1.2.1.** A sequence $\{x_n\} \subset X$ converges to an element $x \in X$ if for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

**Definition 1.2.2.** A sequence in a metric space $(X, d)$ is a Cauchy sequence if for all $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq n_0$.

**Definition 1.2.3.** A metric space $(X, d)$ is called complete if every Cauchy sequence converges in $X$.

### 1.2.1 $b$-Metric Spaces

Czerwik [48] introduced the concept of $b$-metric spaces.

**Definition 1.2.4.** If $M(\neq \phi)$ is a set having $s\ (>= 1) \in \mathbb{R}$ then a self-map $\rho$ on $M$ is called a $b$-metric if the following conditions are satisfied:

**(i)** $\rho(x, y) = 0$ if and only $x = y$;

**(ii)** $\rho(x, y) = \rho(y, x)$;

**(iii)** $\rho(x, z) \leq s.[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in M$.

The pair $(M, \rho)$ is called a $b$-metric space.

From the above definition it is evident that the $b$-metric space extended the metric space. Here, for $s = 1$ it reduces into standard metric space.

### 1.2.2 Cone Metric Spaces

Huang and Zhang [77] considered cone metric spaces, defined convergence and Cauchy sequence in term of interior points of the underlying cone. A subset $P$ of real Banach space $E$ is called an order cone if and only if:

**(1)** $P$ is closed, nonempty and $P \neq \{0\}$,

**(2)** $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ implies $ax + by \in P$,

**(3)** $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial ordering '$\leq$' with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. A cone $P$ is called normal if there is a number $L > 0$ such that for all $x, y \in E$.

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of $P$, while $x \ll y$ stands for $(y - x) \in \text{int } P$.

**Definition 1.2.5.** Let $X$ be a nonempty set of $E$. Suppose that the map $d : X \times X \to E$ satisfies:

**(d1)** $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

**(d2)** $d(x, y) = d(y, x)$ for all $x, y \in X$,

**(d3)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and the pair $(X, d)$ is called a cone metric space.

### 1.2.3 $G$-Metric Spaces

A more appropriate generalization of metric spaces then that of $G$-metric spaces was introduced by Mustafa and Sims [119]. Given below are some definitions and properties for $G$-metric spaces as specified by them.

**Definition 1.2.6.** Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:

**(G1)** $G(x, y, z) = 0$ if $x = y = z$,

**(G2)** $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

**(G3)** $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

**(G4)** $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables and

**(G5)** $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function $G$ is called a generalized metric or more specifically a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

## 1.3 Contractions

A self map $T$ on a metric space $(X, d)$ is Banach contraction if

$$d(Tx, Ty) \leq \; k.d(x, y) \text{ for all } x, y \in X \tag{1.3.1}$$

where $0 \leq k < 1$.

The map $T$ is called nonexpansive if $k = 1$ in (1.3.1) and Lipshitz if $k \geq 0$.

Notice that contraction $\Rightarrow$ nonexpansive $\Rightarrow$ Lipshitz, and the reverse implications are not true.

A self-map $T$ of a metric space is strictly contractive if

$$d(Tx, Ty) < d(x, y) \text{ for all distinct } x, y \in X. \tag{1.3.2}$$

A map $T$ satisfying (1.3.2) need not have a fixed point on a complete metric space. Although Edelstein [60] showed that $T$ satisfying (1.3.2) has a unique fixed point on a compact metric space.

### 1.3.1 Banach Contraction Principle

The Banach contraction principle (BCP) states that "A contraction map of a complete metric space has a unique fixed point."

**Theorem 1.3.1.** *[9] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction. Then $T$ has a unique fixed point in $X$ and for each $x_0 \in X$ the sequence of iterates $\{T^n x_0\}$ converges to the fixed point.*

*Proof.* We select $x_0 \in X$ and define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n \text{ for } n = 0, 1, 2, 3, ....$$

observe that for any indices $n, p \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+p}) &= d(T^n x_0, T^{n+p} x_0) = d(T^n x_0, T^n T^p x_0) \\ &\leq k^n d(x_0, T^p x_0) \\ &\leq k^n [d(x_0, Tx_0) + d(Tx_0, T^2 x_0) + ... + d(T^{p-1} x_0, T^p x_0)] \\ &\leq k^n [1 + k + ... + k^{p-1}] d(x_0, Tx_0) \\ &\leq k^n \frac{1 - k^p}{1 - k} d(x_0, Tx_0). \end{aligned}$$

Which shows that $\{x_n\}$ is a Cauchy sequence and since $X$ is complete there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$. To see that $x$ is a the unique fixed point of $T$ observe that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tx.$$

However, $x = Tx$ and $y = Ty$ imply $d(x, y) = d(Tx, Ty) \leq kd(x, y)$, yielding $d(x, y) = 0$ iff $x = y$. □

The versatility of the applications of the BCP can be judged by the following quote as given by Peitgen *et al.* [130].

*"If the works and achievements of mathematicians could be patented then the contraction mapping principle would probably be among those with the highest earnings up to now and the future."*

### 1.3.2 Kannan Contraction

Let $(X, d)$ be a metric space and $T$ a self map on $X$. Then $T$ is Kannan contraction if the following condition is satisfied

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \tag{1.3.3}$$

where $0 \leq k < \frac{1}{2}$.

Kannan [91] proved that:

**Theorem 1.3.2.** *A map satisfying (1.3.3) has a unique fixed point in a complete metric space.*

A fixed point theorem for discontinuous maps was projected for the first time by

Kannan. Kannan theorem inspired copious extensions and generalizations of the BCP and his own fixed point theorems on various settings [146].

## 1.4 An outline of the work

The basic intent of this thesis is to study the fixed points in different metric spaces. As a result in Chapter 2, fixed point theorems on $b$-metric spaces have been derived. In Chapter 3, the elements of cone metric spaces and its properties have been discussed. In this chapter, fixed point theorems on cone metric spaces have been obtained.

Some fixed point theorems on $G$-metric spaces has been extended in Chapter 4 as well.

The theory of iteration process for computing fixed points for cubic polynomial, and a function in two and three dimensions is developed and placed in Chapter 5. Here the main intent is to offer a comparative study of a few prominent iterative procedures in numerical praxis. The experimental analysis in this chapter seems to be useful to computer programmers and numerical analysts.

There are plenty of problems in applied mathematics which can be solved by means of fixed point theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required. In Chapter 6 we establish some approximate fixed point results in metric spaces under various contraction conditions.

# Chapter 2

# Fixed Point Theorems in $b$-Metric Space

In this chapter we have obtained some fixed point theorems on $b$-metric space which are the extension of fixed point theorems given by Hardy [73] and Reich [140].

## 2.1 Introduction

In the development of non-linear analysis, fixed point theory plays a very important role. Also, it has been widely used in different branches of engineering and sciences. Metric fixed point theory is an essential part of mathematical analysis because of its applications in different areas like variational and linear inequalities, improvement, and approximation theory. The fixed point theorem in metric spaces plays a significant role to construct methods to solve the problems in mathematics and sciences. Although metric fixed point theory is a vast field of study and is capable of solving many equations. To overcome the problem of measurable functions w.r.t. a measure and their convergence, Czerwik [48–51] needs an extension of metric space. Using this idea, he presented a generalization of the renowned Banach fixed point theorem in the $b$-metric spaces. Many researchers studied the $b$-metric space such as Aydi [7], Boriceanu [24–26], Bota [27], Chug [40], Du [194], Kir [102], Olaru [121], Olatinwo [122], Păcurar [133, 134], Rao [138], Roshan [157], Shi [166].

In this chapter, our aim is to show the validity of some important fixed point results into $b$-metric spaces.

## 2.2 Preliminaries

We recall some definitions and properties for $b$-metric spaces given by Czerwik [48].

**Definition 2.2.1.** If $M(\neq \phi)$ is a set having $s\ (>= 1) \in \mathbb{R}$ then a self-map $\rho$ on $M$ is called a $b$-metric if the following conditions are satisfied:

**(i)** $\rho(x, y) = 0$ if and only $x = y$;

**(ii)** $\rho(x, y) = \rho(y, x)$;

**(iii)** $\rho(x, z) \leq s.[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in M$.

The pair $(M, \rho)$ is called a $b$-metric space.

From the above definition it is evident that the $b$-metric space extended the metric space. Here, for $s = 1$ it reduces into standard metric space.

Let us have a look on some example [19] of $b$-metric space:

**Example 2.2.1.** The space $l_p$, $(0 < p < 1)$,

$$l_p = \left\{ (x_n) \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the function $\rho : l_p \times l_p \to \mathbb{R}$ where

$$\rho(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

for all $x = x_n$, $y = y_n \in l_p$ is a $b$-metric space.

**Example 2.2.2.** The space $l_p$, $(0 < p < 1)$, of all real functions $x(t)$, $t \in [0, 1]$ such that

$$\int_0^1 |x(t)|^p \, dt < \infty,$$

is $b$-metric space if we take

$$\rho(x, y) = \left( \int_0^1 |x(t) - y(t)|^p \, dt \right)^{\frac{1}{p}}$$

for each $x, y \in l_p$.

Now we present the definition of Cauchy sequence, convergent sequence and completeness in $b$-metric space.

**Definition 2.2.2.** [48] Let $(M, \rho)$ be a $b$-metric space then $\{x_n\}$ in $M$ is called

(a) a Cauchy sequence iff $\forall \epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$, such that for each $n, m \geq n(\epsilon)$ we have $\rho(x_n, x_m) < \epsilon$.

(b) a convergent sequence if and only if there exist $x \in M$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$, such that for every $n \geq n(\epsilon)$ we have $\rho(x_n, x) < \epsilon$.

**Definition 2.2.3.** [48] 1. If $(M, \rho)$ is a $b$-metric space then a subset $L \subset M$ is called

(i) compact iff for every sequence of elements of $L$ there exists a subsequence that converges to an element of $L$.

(ii) closed iff for each sequence $\{x_n\}$ in $L$ which converges to an element $x$, we have $x \in L$.

2. The $b$-metric space is complete if every Cauchy sequence converges.

To prove the theorem 2.3.2 and 2.3.4 we will use the following lemma 2.2.1 [173].

**Lemma 2.2.1.** Suppose $(M, \rho)$ be a $b$-metric space and $\{y_n\}$ be a sequence in $M$ such that

$$\rho(y_{n+1}, y_{n+2}) \leq \lambda \rho(y_n, y_{n+1}), n = 0, 1, ... \tag{2.2.1}$$

where $0 \leq \lambda < 1$, Then the sequence $\{y_n\}$ is a Cauchy sequence in $M$ provided that $s.\lambda < 1$.

## 2.3 Main Result

The following theorem is given by Reich [140]:

**Theorem 2.3.1.** *Let $M$ be a complete metric space with metric $\rho$ and let $T : M \to M$ be a function with the following property*

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, y)$$

*for all $x, y \in M$, where $a, b, c$ are non-negative and satisfy $a + b + c < 1$. Then $T$ has a unique fixed point.*

We have extended the above theorem (2.3.1) to the $b$-metric space as follows:

**Theorem 2.3.2.** *Let $M$ be a complete $b$-metric space with metric $\rho$ and let $T : M \to M$ be a function with the following*

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, y) \tag{2.3.1}$$

*$\forall x, y \in M$, where $a, b, c$ are non-negative real numbers and satisfy $a + s(b + c) < 1$ for $s \geq 1$ then $T$ has a unique fixed point.*

*Proof.* Let $x_0 \in M$ and $\{x_n\}$ be a sequence in $M$, such that

$$x_n = Tx_{n-1} = T^n x_0$$

Now

$$\begin{aligned}\rho(x_{n+1}, x_n) &= \rho(Tx_n, Tx_{n-1}) \\ &\leq a\rho(x_n, Tx_n) + b\rho(x_{n-1}, Tx_{n-1}) + c\rho(x_n, x_{n-1}) \\ &= a\rho(x_n, x_{n+1}) + b\rho(x_{n-1}, x_n) + c\rho(x_n, x_{n-1})\end{aligned}$$

$$\implies (1-a)\rho(x_{n+1}, x_n) \leq (b+c)\rho(x_n, x_{n-1})$$

$$\implies \rho(x_{n+1}, x_n) \leq \frac{(b+c)}{(1-a)}\rho(x_n, x_{n-1}) = p\rho(x_n, x_{n-1})$$

where $p = \frac{(b+c)}{(1-a)} < \frac{1}{s}$

continuing this process we can easily say that $\rho(x_{n+1}, x_n) \leq p^n \rho(x_0, x_1)$

This implies that $T$ is a contraction mapping.

Now,it is to show that $\{x_n\}$ is a Cauchy sequence in $M$.

Let $m, n > 0$, with $m > n$

$$\rho(x_n, x_m) \leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)]$$

$$\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2})$$

$$+ s^3\rho(x_{n+2}, x_{n+3}) + ...$$

$$\leq sp^n\rho(x_0, x_1) + s^2p^{n+1}\rho(x_0, x_1) + s^3p^{n+2}\rho(x_0, x_1) + ...$$

$$= sp^n\rho(x_0, x_1)\left[1 + sp + (sp)^2 + (sp)^3 + ...\right] = \frac{sp^n}{1-sp}\rho(x_0, x_1)$$

Now using lemma 2.2.1 and taking limit $n \to \infty$ we get

$$\lim_{n\to\infty} \rho(x_n, x_m) = 0$$

$\implies \{x_n\}$ is a Cauchy sequence in $M$.

Since $M$ is complete, we consider that $\{x_n\}$ converges to $x^*$.

Now, we show that $x^*$ is fixed point of $T$.

We have

$$\begin{aligned}
\rho(x^*, Tx^*) &\leq s\,[\rho(x^*, x_n) + \rho(x_n, Tx^*)] \\
&\leq s\,[\rho(x^*, x_n) + \rho(Tx_{n-1}, Tx^*)] \\
&\leq s\,[\rho(x^*, x_n) + a\rho(x^*, Tx^*) + b\rho(x_{n-1}, Tx_{n-1}) + c\rho(x_{n-1}, x^*)] \\
& (1-as)\rho(x^*, Tx^*) \leq s\,[\rho(x^*, x_n) + b\rho(x_{n-1}, x_n) + c\rho(x_{n-1}, x^*)] \\
& \rho(x^*, Tx^*) \leq \frac{s}{(1-as)}\,[\rho(x^*, x_n) + b\rho(x_{n-1}, x_n) + c\rho(x_{n-1}, x^*)] \\
&\leq \frac{s}{(1-as)}\,[\rho(x^*, x_n) + bp^n\rho(x_0, x_1) + c\rho(x_{n-1}, x^*)]
\end{aligned}$$

Taking $\lim n \to \infty$, we get

$$\lim_{n\to\infty} \rho(x^*, Tx^*) = 0$$

$$\implies x^* = Tx^*$$

$\implies x^*$ is the fixed point of $T$. Now, for the uniqueness of fixed point. Let $x$ and $y$ be two fixed points of $T$

$$x = Tx, \;\; y = Ty$$

$$\rho(x,y) = \rho(Tx,Ty) \leq a\rho(x,Tx) + b\rho(y,Ty) + c\rho(x,y)$$

$$\rho(x,y) \leq c\rho(x,y)$$

which is a contradiction. Hence the proof is complete. □

Now we will discuss the extension of the following theorem given by Hardy and Rogers [73] to the $b$-metric space as our second result in theorem 2.3.4.

**Theorem 2.3.3.** *Let $(M,\rho)$ be a metric space and $T : M \to M$ a mapping satisfies the following condition*

*(i) $\rho(Tx,Ty) \leq a.\rho(x,Tx) + b.\rho(y,Ty) + c.\rho(x,Ty) + e.\rho(y,Tx) + f.\rho(x,y)$,*

*where $a,b,c,e,f$ are nonnegative and we set $\alpha = a+b+c+e+f$. Then*

*(a) If $M$ is complete metric space and $\alpha < 1$ then $T$ has a unique fixed point.*

*(b) If (i) is modified to the condition $x \neq y$*

*for all $x,y \in M$ then this implies*

$$\rho(Tx,Ty) \leq a.\rho(x,Tx) + b.\rho(y,Ty) + c.\rho(x,Ty) + e.\rho(y,Tx) + f.\rho(x,y)$$

*and in this case we assume $M$ is compact. $T$ is continuous and $\alpha = 1$, then $T$ has a unique fixed point.*

**Theorem 2.3.4.** *Let* $(M, \rho)$ *be a complete b-metric space and a mapping* $T : M \to M$ *satisfying the following condition for all* $x, y \in M$

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, Ty) + e\rho(y, Tx) + f\rho(x, y) \tag{2.3.2}$$

*where* $a, b, c, e, f$ *are nonnegative and we set* $\alpha = a+b+c+e+f$*, such that* $\alpha \in (0, \frac{1}{2s})$*, for* $s \geq 1$ *then* $T$ *has a unique fixed point.*

Before going to prove this theorem we require following lemma 2.3.1 [73].

**Lemma 2.3.1.** Let the condition 2.3.2 hold on $(M, \rho)$ for a self map $T$ on it. Then if $\alpha \in (0, \frac{1}{2s})$ there exist $\beta < \frac{1}{2s}$ such that

$$\rho(Tx, T^2x) \leq \beta\rho(x, Tx). \tag{2.3.3}$$

*Proof.* Let $y = Tx$ in (2.3.2) and simplify to get

$$\rho(Tx, T^2x) \leq \frac{a+f}{1-b}\rho(x, Tx) + \frac{c}{1-b}\rho(x, T^2x) \tag{2.3.4}$$

Now using triangular inequality $\rho(x, T^2x) \leq s[\rho(x, Tx) + \rho(Tx, T^2x)]$ so from (2.3.4) we obtain

$$\frac{1}{s}\rho(T^2x, x) - \rho(Tx, x) \leq \frac{a+f}{1-b}\rho(x, Tx) + \frac{c}{1-b}\rho(x, T^2x)$$

on simplifying

$$\rho(T^2x, x) \leq \frac{(1+a+f-b)s}{1-b-c.s}\rho(x, Tx) \tag{2.3.5}$$

Now substituting inequality (2.3.5) into (2.3.4), we get

$$\rho(Tx, T^2x) \leq \left(\frac{a + f + c.s}{1 - b - c.s}\right) \rho(x, Tx) \tag{2.3.6}$$

using symmetry, we can exchange $a$ with $b$ and $c$ with $e$ in (2.3.6) to obtain

$$\rho(Tx, T^2x) \leq \left(\frac{b + f + e.s}{1 - b - e.s}\right) \rho(x, Tx) \tag{2.3.7}$$

and then

$$\beta = \min\left(\frac{a + f + c.s}{1 - b - c.s}, \frac{b + f + e.s}{1 - b - e.s}\right) \tag{2.3.8}$$

satisfies the conclusion of this lemma. □

**Proof of Theorem 2.3.4.**

Let $x_0 \in M$ and $\{x_n\}$ be a sequence in $M$, such that

$$x_n = Tx_{n-1} = T^n x_0$$

Using lemma 2.3.1 we can show that

$$\rho(x_{n+1}, x_n) \leq \beta^n \rho(x_0, x_1) \tag{2.3.9}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in $M$.

Let $m, n > 0$, with $m > n$

$$\rho(x_n, x_m) \leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)]$$

$$\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2})$$

$$+ s^3\rho(x_{n+2}, x_{n+3}) + ...$$

$$\leq s\beta^n\rho(x_0, x_1) + s^2\beta^{n+1}\rho(x_0, x_1)$$

$$+ s^3\beta^{n+2}\rho(x_0, x_1) + ...$$

On taking $\lim n \to \infty$ we get $\lim_{n\to\infty} \rho(x_n, x_m) = 0$.

This implies $\{x_n\}$ is a Cauchy sequence in $M$.

Since $M$ is complete, we consider that $\{x_n\}$ converges to $x^*$.

Now, to show that $x^*$ is fixed point of $T$.

we have

$$\rho(x^*,Tx^*) \leq s\,[\rho(x^*,x_n)+\rho(x_n,Tx^*)]$$

$$\leq s\,[\rho(x^*,x_n)+\rho(Tx_{n-1},Tx^*)]$$

$$\leq s[\rho(x^*,x_n)+a\rho(x_{n-1},Tx_{n-1})$$

$$+\,b\rho(x^*,Tx^*)+c\rho(x_{n-1},Tx^*)$$

$$+\,e\rho(x^*,Tx_{n-1})+f\rho(x_{n-1},x^*)]$$

$$\implies \rho(x^*,Tx^*) \leq s[a\rho(x_{n-1},x_n)+b\rho(x^*,Tx^*)$$

$$+\,c\rho(x_{n-1},Tx^*)+(e+1)\rho(x^*,x_n)+f\rho(x_{n-1},x^*)]$$

Taking $\lim n \to \infty$, we get, $\rho(x^*,Tx^*) \leq s(b+c)\rho(x^*,Tx^*)$

which is contradiction unless $x^* = Tx^*$.

Now to show the uniqueness of fixed point. Let us consider $x$ and $y$ be two fixed points of $T$, sothat $x = Tx$ , $y = Ty$.

Again

$$\rho(x,y) = \rho(Tx,Ty) \leq a\rho(x,Tx) + b\rho(y,Ty) + c\rho(x,Ty)$$

$$+ e\rho(y,Tx) + f\rho(x,y)$$

$$\leq (c+e+f)\rho(x,y)$$

which is a contradiction.

Hence the proof is complete.

## 2.4 Conclusions

In this chapter we have attained some fixed point results in complete $b$-metric space which are the extension of the theorems given by Reich and Hardy-Rogers for complete metric space.

# Chapter 3

# Fixed Point Theorems in Cone Metric Space

Some fixed point theorems on cone metric spaces have been derived in this chapter. We have studied the $T$-contraction and obtained the extension of the fixed point theorems on compatible maps given by Gerald Jungck [86] in cone metric spaces.

## 3.1 Introduction

Ordered spaces and cones have applications in mathematical and other sciences. $K$-metric and $K$-normed spaces were first studied by Kantorovitch [92], Vandergraft

[190], Zaberejko [197] and Aliprantis [3] using an ordered Banach space instead of the set of real numbers. Further, Lin [109] considered the notion of $K$-metric spaces by replacing real numbers with cone $P$ in the metric function, $d : X \times X \to K$.

Huang and Zhang [77] considered such spaces under the name of cone metric spaces, defined convergence and Cauchy sequence in terms of interior points of the underlying cone. Huang, Zhang and other researchers derived some fixed point and common fixed point theorems for contractive-type mappings in cone metric spaces (See for an instance [1], [13], [14], [38], [77], [79], [94], [95], [100], [108], [135], [141], [159], [174], [185] and [186]).

## 3.2 Preliminaries

Throughout this section $\mathbb{R}^+$, denotes the set of all nonnegative real numbers, $E$, a real Banach space and $\mathbb{N}$, the set of natural numbers.

The following definitions and lemmas are due to Huang and Zhang [77].

A subset $P$ of $E$ is called an order cone if and only if:

**(1)** $P$ is closed, nonempty and $P \neq \{0\}$;

**(2)** $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

**(3)** $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial ordering '$\leq$' with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. A cone $P$ is called normal if there is a number $L > 0$ such that for all $x, y \in E$.

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number $L$ satisfying the above inequality is called the normal constant of $P$, while $x \ll y$ stands for $(y - x) \in \text{int } P$.

**Definition 3.2.1.** Let $X$ be a nonempty set of $E$. Suppose that the map $d : X \times X \rightarrow E$ satisfies:

**(d1)** $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

**(d2)** $d(x, y) = d(y, x)$ for all $x, y \in X$,

**(d3)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and the pair $(X, d)$ is called a cone metric space.

**Definition 3.2.2.** Let $(X,d)$ be a cone metric space. We say that $\{x_n\}$ is

**(a)** a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is an $N$ such that $d(x_n, x_m) \ll c$ for all $n, m > N$.

**(b)** a convergent sequence if for every $c \in E$ with $0 \ll c$, there is an $N$ such that $d(x_n, x) \ll c$ for all $n, m > N$ and for some fixed $x \in X$.

**Definition 3.2.3.** A cone metric space $X$ is said to be complete if for every Cauchy sequence in $X$ is convergent in $X$. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided that $P$ is a normal cone with normal constant $L$.

**Lemma 3.2.1.** Let $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $L$. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $d(x_n, x) \to 0 \;\; (as \;\; n \to \infty)$.

**Definition 3.2.4.** Let $(X,d)$ be a cone metric space. If for any sequence $\{x_n\}$ in $X$, there exist a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ then $(X,d)$ is called a sequentially compact cone metric space.

Now we are extending the definition of asymptotic regularity [160] to the cone metric space.

**Definition 3.2.5.** Let $f$, $g$ and $h$ be self maps on a cone metric space $(X, d)$. The pair $(f, g)$ is asymptotically regular with respect to $h$ at $x_0 \in X$ if there exists a sequence $\{x_n\}$ in $X$ such that

$$hx_{2n+1} = fx_{2n},\ hx_{2n+2} = gx_{2n+1},\ n = 0, 1, 2, ...,$$

$$\text{and } \lim_{n\to\infty}\ d(hx_n, hx_{n+1}) = 0.$$

**Definition 3.2.6.** Let $(X, d)$ be a cone metric space and $T, S : X \to X$ be two functions. A mapping $S$ is said to be a $T$-contraction if there exists $k \in (0, 1)$ such that for

$$d(TSx, TSy) \leq kd(Tx, Ty) \ \ \forall x, y \in X. \tag{3.2.1}$$

Asymptotically $T$-regular maps was introduced in [17], we are extending this definition to the cone metric space.

**Definition 3.2.7.** Let $S$ and $T$ be two self-maps on cone metric space $(X, d)$ and $\{x_n\}$ a sequence in $X$. Then $\{x_n\}$ is said to be asymptotically $T$-regular with respect to $S$ if $d(Sx_n, Tx_n) \to 0$ as $n \to \infty$.

**Definition 3.2.8.** Let $(X, d)$ be a cone metric space. A mapping $T : X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ is also convergent.

**Definition 3.2.9.** Let $(X, d)$ be a cone metric space. A mapping $T : X \to X$ is said subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

**Proposition 3.2.1.** *If $(X, d)$ be a compact cone metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

## 3.3 Fixed Point Theorems in Cone Metric Space

**Theorem 3.3.1.** *Let $T,\ S$ be a one-to-one, continuous and subsequentially convergent self map on a complete cone metric space $(X, d)$. Then for every $T$-contractive continuous map $S$ has a unique fixed point. Also if $T$ is sequentially convergent, then for each $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

*Proof.* Let $x_1, x_2 \in X$,

$$\begin{aligned} d(Tx_1, Tx_2) &\leq d(Tx_1, TSx_1) + d(TSx_1, TSx_2) + d(TSx_2, Tx_2) \\ &\leq d(Tx_1, TSx_1) + kd(Tx_1, Tx_2) + d(TSx_2, Tx_2) \end{aligned}$$

so

$$d(Tx_1, Tx_2) \leq \frac{1}{1-k}[d(Tx_1, TSx_1) + d(TSx_2, Tx_2)] \tag{3.3.1}$$

On selecting $x_0 \in X$ and defining the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, 3, ....$ By (3.3.1) for any indices $m, n \in N$,

$$\begin{aligned} d(Tx_n, Tx_m) &= d(TS^n x_0, TS^m x_0) \\ &\leq \frac{1}{1-k}[d(TS^n x_0, TS^{n+1} x_0) + d(TS^{m+1} x_0, TS^m x_0)] \\ &\leq \frac{1}{1-k}[k^n d(Tx_0, TSx_0) + k^m d(TSx_0, Tx_0)] \end{aligned}$$

hence

$$d(TS^n x_0, TS^m x_0) \leq \frac{k^n + k^m}{1-k}[d(Tx_0, TSx_0)] \qquad (3.3.2)$$

Above inequality (3.3.2) and condition $0 < k < 1$ show that $\{TS^n x_0\}$ is a Cauchy sequence and since $X$ is complete there exists $a \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = a \qquad (3.3.3)$$

Since $T$ is subsequentially convergent $\{S^n x_0\}$ has a convergent subsequence. So, there exist $b \in X$ and $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{n\to\infty} S^{n_k} x_0 = b$$

hence

$$\lim_{n\to\infty} TS^{n_k} x_0 = Tb$$

and by (3.3.3), we conclude that

$$Tb = a \qquad (3.3.4)$$

Since $S$ is continuous and $\lim\limits_{n\to\infty} S^{n_k} x_0 = b$ then $\lim\limits_{n\to\infty} S^{n_k+1} x_0 = Sb$ and so

$$\lim_{n\to\infty} TS^{n_{k+1}}x_0 = TSb.$$

Again by (3.3.3), $\lim_{n\to\infty} TS^{n_{k+1}}x_0 = a$ and therefore $TSb = a$. Since $T$ is one-to-one and by (3.3.4), $Sb = b$. Therefore $b$ is a fixed point of $S$.

Since $T$ is one-to-one and $S$ is $T$-contraction, $S$ has a unique fixed point. □

**Corollary 3.3.1.** *Let $(X, d)$ be a complete cone metric space and $S : X \to X$ be a contractive mapping. Then $S$ has a unique fixed point in $X$, and for each $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

*Proof.* In the above theorem (3.3.1) on taking $Tx = x$ ($T$ as identity function), we can conclude the proof of corollary. □

**Theorem 3.3.2.** *Let $(X, d)$ be a compact cone metric space and $T : X \to X$ be a one-to-one and continuous mapping. Then for every $T$-contractive mapping $S : X \to X$, $S$ has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

*Proof.* To prove above theorem first we will show that $S$ is continuous.

Let $\lim_{n\to\infty} x_n = x$.

Since $S$ is $T$-contractive

$$d(TSx_n, TSx) \leq d(Tx_n, Tx)$$

and due to continuity of $T$ this shows that

$$\lim_{n\to\infty} TSx_n = TSx$$

Let $\{Sx_{n_k}\}$ be an arbitrary convergent subsequence of $\{Sx_n\}$.

There exists a $y \in X$ such that

$$\lim_{n\to\infty} Sx_{n_k} = y.$$

Again by the continuity of $T$,

$$\lim_{n\to\infty} TSx_{n_k} = Ty.$$

By,

$$\lim_{n\to\infty} TSx_n = TSx,$$

we conclude that $TSx = Ty$. Since $T$ is one-to-one so $Sx = y$.

Hence, every convergent subsequence of $\{Sx_n\}$ converges to $Sx$.

Now since $X$ is a compact cone metric space and $S$, $T$ and continuous map. The the map

$$\varphi : X \to [0, +\infty)$$

defined by

$$\varphi(y) = d(TSy, Ty)$$

is continuous on $X$ and hence by compactness of $X$ it attains its minimum, say at $x \in X$. If $Sx \neq x$ then

$$\varphi(Sx) = d(TS^2x, TSx) < d(TSx, Tx)$$

which is a contradiction, so $Sx = x$.

Now let $x_0 \in X$ and set $a_n = d(TS^n x_0, Tx)$. Since

$$a_{n+1} = d(TS^{n+1}x_0, Tx) = d(TSS^n x_0, TSx) \leq d(TS^n x_0, Tx) = a_n$$

then $\{a_n\}$ is a non-increasing sequence of nonnegative real numbers and so has a limit, say $a$.

By compactness, $\{TS^n x_0\}$ has a convergent subsequence $\{TS^{n_k}x_0\}$, say

$$\lim_{n\to\infty} TS^{n_k}x_0 = z, \text{ for } z \in X. \tag{3.3.5}$$

Since $T$ is sequentially convergent then for $w \in X$ we have

$$\lim_{n\to\infty} TS^{n_k}x_0 = w \tag{3.3.6}$$

By (3.3.5) and (3.3.6), $Tw = z$. So $d(Tw, Tx) = a$.

Now we show that $Sw = x$. If $Sw \neq x$, then

$$\begin{aligned} a = \lim_{n\to\infty} d(TS^n x_0, Tx) &= \lim_{n\to\infty} d(TS^{n_k}x_0, Tx) \\ &= d(TSw, Tx) \\ &= d(TSw, TSx) \\ &< d(Tw, Tx) = a \end{aligned}$$

that is contradiction. So $Sw = x$ and hence,

$$\lim_{n\to\infty} d(TS^{n_{k+1}}x_0, Tx) = d(TSw, Tx) = 0$$

Therefore, $\lim_{n\to\infty} d(TS^n x_0, Tx) = Tx_0$.

Since $T$ is sequentially convergent (by Proposition 3.2.1), then $\lim_{n\to\infty} d(S^n x_0, Tx) = x$. □

**Corollary 3.3.2.** *Let $(X, d)$ be a compact cone metric space and $S : X \to X$ be a contractive mapping. Then $S$ has a unique fixed point in $X$, and for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

*Proof.* Proof is obvious from the above if we take $T$ as identity function i.e $Tx = x$. □

In 1986, Gerald Jungck [86] have introduced definition of compatible maps on metric space $(X, d)$. Now to introduce our next result, we are extending the concept of compatible maps to cone metric space.

**Definition 3.3.1.** Let $(X, d)$ be a cone metric space, the self maps $S$ and $T$ on $X$, is said to be compatible iff

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

Now to derive our next result (3.3.3), we need to extend the proposition [86] to the cone metric space.

**Proposition 3.3.1.** *Let $S$ and $T$ be compatible self maps on a cone metric space $(X, d)$.*

**(1)** *If $St = Tt$, then $STt = TSt$.*

**(2)** *Suppose that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $x_n \in X$.*

  **(a)** *If $S$ is continuous at $t$, $\lim_{n\to\infty} TSx_n = St$.*

  **(b)** *If $S$ and $T$ are continuous at $t$, then $St = Tt$ and $STt = TSt$.*

**Theorem 3.3.3.** *Let $S, T, U$ be three self maps on a cone metric space $(X, d)$ satisfying following conditions:*

**(1)** $d(Tx, Uy) < a_1 d(Tx, Sx) + a_2 d(Uy, Sy) + a_3 d(Tx, Sy) + a_4 d(Uy, Sx) + a_5 d(Sx, Sy)$, *for $x, y \in X$, where each $a_i > 0$ and $\max\{a_2 + a_4, a_3 + a_4 + a_5\} < 1$,*

**(2)** *$S$ is continuous*

**(3)** *$(S, T)$ and $(S, U)$ are compatible pairs, and*

**(4)** *There exist a sequence $\{x_n\}$ in $X$ which is asymptotically $T$-regular as well as $U$-regular with respect to $S$.*

*Then $S$, $T$, and $U$ have a unique common fixed point.*

*Proof.*

$$
\begin{aligned}
d(Sx_n, Sx_m) &\leq d(Sx_n, Tx_n) + d(Tx_n, Ux_m) + d(Ux_m, Sx_m) \\
&\leq d(Sx_n, Tx_n) + a_l d(Tx_n, Sx_n) + a_2 d(Ux_m, Sx_m) + a_3 d(Tx_n, Sx_m) \\
&+ a_4 d(Ux_m, Sx_n) + a_5 d(Sx_n, Sx_m) + d(Ux_m, Sx_m)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
d(Sx_n, Sx_m) \leq &\left(\frac{1 + a_1 + a_3}{1 - a_3 - a_4 - a_5}\right) d(Sx_n, Tx_n) \\
&+ \left(\frac{1 + a_2 + a_4}{1 - a_3 - a_4 - a_5}\right) d(Ux_m, Sx_m)
\end{aligned}
$$

This shows that $\{Sx_n\}$ is a Cauchy sequence.

Put $\lim_{n \to \infty} Sx_n = z$, $z \in X$.

Then it follows from (3) that $\lim_{n \to \infty} Tx_n = z$ and $\lim_{n \to \infty} Ux_n = z$.

Also by (2), we find that $Sx_n \to z$, $Tx_n \to z$, $S^2 x_n \to Sz$, $STx_n \to Sz$ and $SUx_n \to z$.

Now by proposition (3.3.1) $TSx_n \to Sz$, since $\{S, T\}$ is a compatible pair. Similarly, we conclude that $SUx_n \to Sz$ and $USx_n \to Sz$.

$$
\begin{aligned}
d(Sz, Uz) &\leq d(Sz, TSx_n) + d(TSx_n, Uz) \\
&\leq d(Sz, TSx_n) + a_l d(TSx_n, S^2 x_n) + a_2 d(Uz, Sz) \\
&+ a_3 d(TSx_n, Sz) + a_4 d(Uz, S^2 x_n) + a_5 d(S^2 x_n, Sz)
\end{aligned}
$$

We know that $S^2x_n \to Sz$ and $TSx_n \to Sz$ as $n \to \infty$,

$$\begin{aligned} d(Sz, Uz) &\leq d(Sz, Sz) + a_1 d(Sz, Sz) + a_2 d(Uz, Sz) \\ &\quad + a_3 d(Sz, Sz) + a_4 d(Uz, Sz) + a_5 d(Sz, Sz) \\ &\leq (a_2 + a_4) d(Uz, Sz) \end{aligned}$$

So $Sz = Uz$.

Similarly we can show that $Sz = Tz$. Hence $Sz = Tz = Uz$.

Now consider

$$\begin{aligned} d(Tx_n, Uz) &\leq a_1 d(Sx_n, Tx_n) + a_2 d(Sz, Uz) + a_3 d(Sz, Tx_n) \\ &\quad + a_4 d(Sx_n, Uz) + a_5 d(Sx_n, Sz) \end{aligned}$$

On letting $n \to \infty$, we get

$$d(z, Uz) \leq (a_3 + a_4 + a_5) d(z, Uz)$$

giving there by $z = Uz$. Thus $z$ is a common fixed point of $U$ and $Sz = Tz = Uz$ implies that $z$ is common fixed point for $S$ and $T$.

In order to prove the uniqueness of common fixed point, let $z_1$ and $z_2$ be any two distinct common fixed point of $S$, $T$ and $U$.

$$\begin{aligned} d(z_1, z_2) &= d(Tz_1, Uz_2) \\ &\leq a_1 d(Tz_1, Sz_1) + a_2 d(Uz_2, Sz_2) + a_3 d(Tz_1, Sz_2) \\ &+ a_4 d(Uz_2, Sz_1) + a_5 d(Sz_1, Sz_2) \\ &= (a_3 + a_4 + a_5) d(z_1, z_2) \end{aligned}$$

Which implies $z_1 = z_2$. This completes the proof. □

## 3.4 Conclusions

In this chapter some fixed point theorems in cone metric spaces have been derived for:

- $T$-contraction
- Compatible maps
- Asymptotically regular maps

# Chapter 4

# Some Fixed Point Theorems in $G$-Metric Space

In this chapter the well known notion of a cyclic contraction for a finite family of non-empty subsets has been extended for $G$-metric space $X$.

## 4.1 Introduction

Throughout this chapter $\mathbb{N}$ denotes the set of natural numbers and $\Phi$ the class of the functions $\varphi : [0, 1) \to [0, 1)$ satisfying:

**(a)** $\varphi$ is continuous and monotone nondecreasing,

**(b)** $\varphi(t) = 0$, $t = 0$.

The function $\varphi \in \Phi$ is also known as altering distance function (c.f. [101]).

Herein this chapter derived results are the generalization of recent fixed point theorems of [97], [106], [118].

## 4.2 Cyclic Contraction

Kirk et al. [106] introduced the following notion of cyclic mappings and obtained a fixed point theorem (see Theorem 4.2.1 below).

**Definition 4.2.1.** Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space $(X, d)$. A mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a cyclic mapping (or $p$-cyclic mapping) if

$$T(A_i) \subseteq A_{i+1}, \quad \textit{where} \;\; A_{p+1} = A_1 \;\; \textit{for} \;\; i = 1, 2, ..., p.$$

**Theorem 4.2.1.** *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ cyclic mapping. Assume that there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x \in A_i \textit{ and } y \in A_{i+1}.$$

*Then $T$ has a unique fixed point.*

We refer to [12, 53, 63, 96–99, 106, 120, 178] and references thereof for a detailed study of cyclic mappings.

Recently, Karapinar et.al. [97] (see also [96]) combined the ideas of $(\psi, \varphi)$- weakly contractions, and cyclic contractions and introduced the notion of cyclic weak $(\psi, \varphi)$-contraction as follows:

**Definition 4.2.2.** Let $A_1, A_2, ...A_p$ be nonempty subsets of a metric space $(X, d)$ such that $X = \bigcup_{i=1}^{p} \{A_i\}$. A mapping $T : X \to X$ is said to be cyclic weak $(\psi, \varphi)$-contraction if

1. $X = \bigcup_{i=1}^{p} \{A_i\}$ is a cyclic representation of $X$ with respect to $T$,
2. $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$ for all $x \in A_i$ and $y \in A_{i+1}$,

where $\psi, \varphi \in \Phi$ and $A_{p+1} \subseteq A_1$.

Following theorem is the main result in [97].

**Theorem 4.2.2.** *Let $(X, d)$ be a metric space and $A_1, A_2, ..., A_p$ nonempty closed subsets of $X$ such that $X = \bigcup_{i=1}^{p} \{A_i\}$. Let $T : X \to X$ be a cyclic weak $(\psi, \varphi)$-contraction. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

Before spreading the above theorems in a $G$-metric space, Let us recall some definitions, propositions and properties of $G$-metric spaces given by Mustafa and Sims

[119].

## 4.3 $G$-metric space

**Definition 4.3.1.** Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:

**(G1)** $G(x, y, z) = 0 \; if \; x = y = z$,

**(G2)** $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

**(G3)** $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

**(G4)** $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables and

**(G5)** $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function $G$ is called a generalized metric, or, more specifically, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Definition 4.3.2.** Let $(X, G)$ be a $G$-metric space, and let $\{x_n\}$ a sequence of points in $X$, a point $x$ in $X$ is said to be the limit of the sequence $\{x_n\}$ if

$$\lim_{m,n \to 0} G(x, x_n, x_m) = 0.$$

Then we can says that sequence $\{x_n\}$ is $G$-convergent to $x$.

**Proposition 4.3.1.** *Let $(X, G)$ be a $G$-metric space. Then the following are equivalent:*

**(i)** $\{x_n\}$ *is $G$-convergent to $x$,*

**(ii)** $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

**(iii)** $G(x_n, x, x) = 0$ *as* $n \to \infty$,

**(iv)** $G(x_m, x_n, x) = 0$ *as* $m, n \to \infty$.

**Definition 4.3.3.** Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called $G$-Cauchy if, for each $\varepsilon > 0$, there exist a positive integer $N$ such that

$$G(x_m, x_n, x_l) < \varepsilon,$$

for all $n, m, l \geq N$.

**Proposition 4.3.2.** *Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

**Definition 4.3.4.** A $G$-metric space $(X, G)$ is said to be $G$-complete if every $G$-Cauchy Sequence in $(X, G)$ is $G$-convergent in $X$.

**Proposition 4.3.3.** *Let $(X, G)$ be a $G$-metric space. Then for $x, y, z \in X$ it follows that:*

**(i)** *If* $G(x, y, z) = 0$ *then* $x = y = z$,

**(ii)** $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,

**(iii)** $G(x, y, y) \le 2G(y, x, x)$,

**(iv)** $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,

**(v)** $G(x, y, z) \le \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

**(vi)** $G(x, y, z) \le (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

## 4.4 Fixed Point Results

First we extend the Definition 4.2.1 to $G$-metric space.

**Definition 4.4.1.** Let $A_1, A_2, ..., A_p$ be nonempty subsets of a $G$-metric space $(X, G)$. A mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a cyclic mapping on $G$-metric space, if

$$T(A_i) \subseteq A_{i+1}, \quad \textit{where} \;\; A_{p+1} = A_1 \;\; \textit{for} \;\; i = 1, 2, ..., p.$$

Now we extend the Theorem 4.2.1 on $G$-metric space.

**Theorem 4.4.1.** *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a $G$-complete $G$-metric space $(X, G)$, at least one of which is compact and suppose $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a cyclic mapping, such that*

$$G(Tx, Tx, Ty) \le kG(x, x, y) \text{ for all } x \in A_i,\ y \in A_{i+1} \textit{ and } k \in (0, 1).$$

*Then $T$ has a unique fixed point.*

*Proof.* Let $A_1$ is compact and

$$d = dist(A_1, A_p) := \inf\{G(x, x, y) : x \in A_1, y \in A_p\}.$$

Now by compactness of $A_1$ there exists $x_0 \in A_1$ and a sequence $\{u_n\} \in A_p$ such that

$$\lim_{n\to\infty} G(x_0, x_0, u_n) = d.$$

Since $d > 0$, then

$$G(T^{p+1}x_0, T^{p+1}x_0, T^{p+1}u_n) < ... < G(Tx_0, Tx_0, Tu_n) < G(x_0, x_0, u_n). \qquad (4.4.1)$$

Since the sequence $\left\{T^{\{p+1\}}(u_n)\right\}_{n=1}^{\infty} \subset A_1$ and $A_1$ is compact, this sequence has a subsequence that converges to some $z \in A_1$.

Now by (4.4.1) and continuity of the distance function it must be the case that

$$G(z, z, T^{p+1}x_0) \leq d.$$

However this implies

$$G(T^{p-1}z, T^{p-1}z, T^{2p}x_0) \leq d.$$

Since $T^{p-1}z \in A_p$ and $T^{2p}x_0 \in A_1$ we have a contradiction. We conclude therefore that $d = 0$ and $A_1 \cap A_p \neq \phi$. Thus $A_1 \cap A_2 \neq \phi$.

We now consider the sets $A_1' = A_1 \cap A_2$, $A_2' = A_2 \cap A_3$, $\quad ,A_P' = A_P \cap A_1$. In view of theorem these sets are all nonempty (and closed) and $A_1'$ is compact. Thus conditions of the theorem hold for $T$ and the family $\{A_i'\}_{i=1}^p$, and by repeating the argument just given we conclude

$$A_1' \cap A_p' \neq \phi.$$

This in turn implies $A_1 \cap A_2 \cap A_3 \neq \phi$. Continuing step-by-step we conclude

$$A := \bigcap_{i=1}^{p} A_i = \phi.$$

Since $A$ is compact and the restriction of $T$ to $A$ is contractive, we conclude that $T$ has a unique fixed point in $A$. Uniqueness follows from the fact that any fixed point of $T$ necessarily lies in $A$. $\square$

To present our next result we need the extension of the definition 4.2.2 to $G$-metric space.

**Definition 4.4.2.** Let $A_1, A_2, ...A_p$ be nonempty subsets of a $G$-metric space $(X, G)$ such that $X = \bigcup_{i=1}^{p} \{A_i\}$. A mapping $T : X \to X$ is said to be cyclic weak $(\psi, \varphi)$-contraction if

1. $X = \bigcup_{i=1}^{p} \{A_i\}$ is a cyclic representation of $X$ with respect to $T$,
2. $\psi(G(Tx, Tx, Ty)) \leq \psi(Gx, x, y) - \varphi(Gx, x, y)$ for all $x \in A_i$ and $y \in A_{i+1}$.

where $\psi, \varphi \in \Phi$ and $A_{p+1} \subseteq A_1$.

Following theorem is an extension of Theorem 4.2.2 to $G$-metric space.

**Theorem 4.4.2.** *Let $(X, G)$ be a complete $G$-metric space and $A_1, A_2, ..., A_p$ nonempty closed subsets of $X$ such that $X = \bigcup_{i=1}^{p} \{A_i\}$. Let $T : X \to X$ be a cyclic weak $(\psi, \varphi)$-contraction. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

*Proof.* Take $x_0 \in X$ and consider the sequence given by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, ....$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then, since $x_{n_0+1} = Tx_{n_0} = x_{n_0}$, the part of existence of the fixed point is proved. Suppose that $x_{n+1} \neq x_n$ for any $n = 0, 1, 2, ....$ Then, since $X = \bigcup_{i=1}^{p} A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, ..., m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since $T$ is a cyclic $(\psi, \varphi)$-contraction, we have

$$\psi(G(x_n, x_n, x_{n+1})) = \psi(G(Tx_{n-1}, Tx_{n-1}, Tx_n)).$$

$$\leq \psi(G(x_{n-1}, x_{n-1}, x_n)) - \varphi(G(x_{n-1}, x_{n-1}, x_n)) \leq \psi(G(x_{n-1}, x_{n-1}, x_n)). \quad (4.4.2)$$

From (4.4.2) and on taking into account that $\varphi$ is nondecreasing we obtain

$$G(x_n, x_n, x_{n+1}) \leq G(x_{n-1}, x_{n-1}, x_n) \ \textit{for} \ \ \textit{any}\ n = 1, 2, ....$$

Thus $\{G(x_n, x_n, x_{n+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Consequently, there exists $\lambda \geq 0$ such that $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = \lambda$. Taking $n \to \infty$ in (4.4.2) and using the continuity of $\psi$ and $\varphi$, we have

$$\psi(\lambda) \leq \psi(\lambda) - \varphi(\lambda),$$

and, therefore, $\psi(\lambda) = 0$. Since $\psi \in F$, $\lambda = 0$, that is,

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{4.4.3}$$

In the sequel, we will prove that $\{x_n\}$ is a $G$-Cauchy sequence.

Let

$$G(x_p, x_p, x_q) \leq \frac{\epsilon}{2}. \tag{4.4.4}$$

Since $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0$ we also find $n_1 \in \mathbb{N}$ such that

$$G(x_n, x_n, x_{n+1}) \leq \frac{\epsilon}{2}. \tag{4.4.5}$$

for any $n \geq n_1$.

Suppose that $a, b \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $j \in \{1, 2, ..., m\}$ such that $b - a \equiv j$. Therefore, $b - a + k \equiv 1$ for $k = m - j + 1$. So, we have,

$$G(x_a, x_a, x_b) \leq G(x_a, x_a, x_{b+k}) + G(x_{a+k}, x_{a+k}, x_{b+k-1}) + ... + G(x_{b+1}, x_{b+1}, x_b). \tag{4.4.6}$$

Now By (4.4.4), (4.4.5) and (4.4.6), we get

$$G(x_a, x_a, x_b) \leq \frac{\epsilon}{2} + k\frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m\frac{\epsilon}{2m} = \epsilon. \tag{4.4.7}$$

This proves that $\{x_n\}$ is a $G$-Cauchy sequence. Since $X$ is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

No we will show that $x$ is a fixed point of $T$. Since $X$ is a cyclic representation with respect to $T$, then the sequence $\{x_n\}$ has infinite terms in each $A_i$ for $i \in \{1, 2, ..., m\}$.

Let $x \in A_i$, $Tx \in A_{i+1}$ and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$, then by the use of contractive condition, we can obtain,

$$\psi(G(x_{n_{k+1}}, x_{n_{k+1}}, Tx)) = \psi(G(Tx_{n_k}, Tx_{n_k}, Tx)),$$

$$\leq \psi(G(Tx_{n_k}, Tx_{n_k}, x)) - \varphi(G(x_{n_k}, x_{n_k}, x)) \leq \psi(G(x_{n_k}, x_{n_k}, x)), \qquad (4.4.8)$$

and since $x_{n_k} \to x$ and $\varphi$ and $\psi$ belong to $T$, letting $k \to \infty$ in the (4.4.8), we have

$$\psi(G(x, x, Tx)) \leq \psi(G(x, x, x)) = \psi(0) = 0,$$

or, equivalently, $\psi(G(x, x, Tx)) = 0$. Therefore $x$ is a fixed point of $T$.

Finally, to prove the uniqueness of the fixed point, we have $y, z \in X$ with $y$ and $z$ as fixed points of $T$. The cyclic character of $T$ and the fact that $y, z \in X$ are fixed points of $T$, imply that $y, z \in \bigcap_{i=1}^{m} A_i$. Using the contractive condition we obtain

$$\psi(G(y, y, z)) \leq \psi(G(Ty, Ty, Tx)),$$

$$\leq \psi(G(y, y, z)) - \varphi(G(y, y, z)) \leq \psi(G(y, y, z)), \qquad (4.4.9)$$

hence $\psi(G(y, y, z)) = 0$ thus $G(y, y, z) = 0$ and consequently $y = z$. This prove the uniqueness of the fixed point. □

Now to extend the theorem Theorem 2.3 [118] as our next result we need the following definition in $G$-metric space.

**Definition 4.4.3.** Let $A_1, A_2, ..., A_p$ be nonempty subsets of a $G$-metric space $(X, G)$.

A cyclic mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ will be called a generalized cyclic weak $(\psi, \varphi)$-contraction if

$$\psi(G(Tx,Tx,Ty)) \leq \psi(M(x,x,y)) - \varphi(M(x,x,y)) \tag{4.4.10}$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\psi, \varphi \in \Phi$, $A_{p+1} = A_1$ and $M(x,x,y) = \max\{G(x,x,y), G(x,x,Tx), G(y,y,Ty), \frac{G(x,x,Ty)+G(y,y,Tx)}{2}\}$.

**Theorem 4.4.3.** *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete $G$-metric space $(X,G)$ and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a generalized cyclic weak mapping on $X$. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

*Proof.* Suppose for some $i \in \{1, 2, ..., p\}$ there exists an $x \in A_i$ satisfying (4.4.10). Since for any $n \in \mathbb{N}$, either $n$ or $n+1$ is even, we have

$$\begin{aligned} \psi(G(T^n x, T^n x, T^{n+1} x)) &\leq \psi(M(T^{n-1}x, T^{n-1}x, T^n x)) - \\ &\quad \varphi(M(T^{n-1}x, T^{n-1}x, T^n x)). \qquad (4.4.11) \\ &\leq \psi(M(T^{n-1}x, T^{n-1}x, T^n x). \end{aligned}$$

Since $\psi$ is nondecreasing, we have

$$\begin{aligned} G(T^n x, T^n x, T^{n+1} x) &\leq \max\{G(T^{n-1}x, T^{n-1}x, T^n x), G(T^{n-1}x, T^{n-1}x, T^n x), \\ G(T^n x, T^n x, T^{n+1} x), &\frac{G(T^{n-1}x, T^{n-1}x, T^{n+1}x) + G(T^{n+1}x, T^{n+1}x, T^{n+1}x)}{2}\} \\ &\leq G(T^{n-1}x, T^{n-1}x, T^n x), \end{aligned}$$

for $n \in \mathbb{N}$. Thus $G(T^n x, T^n x, T^{n+1}x)$ is a decreasing sequence of nonnegative real numbers. If $\lim_{n\to\infty} G(T^n x, T^n x, T^{n+1}x) = r$ for some $r > 0$. Making $n \to \infty$ in (4.4.11) and using the continuity of $\psi$ and $\varphi$, we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which is a contradiction. Hence

$$\lim_{n\to\infty} G(T^n x, T^n x, T^{n+1}x) = 0$$

Now we show that $\{T^n x\}$ is a $G$-Cauchy sequence.

Suppose $\{T^n x\}$ is not $G$-Cauchy, then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \leq m_k < n_k$,

$$G(T^{m_k}x, T^{m_k}x, T^{n_k}x) \geq \mu$$

and

$$G(T^{m_k}x, T^{m_k}x, T^{n_k-1}x) < \mu.$$

By the triangle inequality,

$$G(T^{m_k}x, T^{m_k}x, T^{n_k}x) \leq G(T^{m_k}x, T^{m_k}x, T^{n_k-1}x) + G(T^{n_k-1}x, T^{n_k-1}x, T^{n_k}x).$$

It follows that

$$G(T^{m_k}x, T^{m_k}x, T^{n_k}x) = \mu.$$

Now by (4.4.10), we have

$$\psi(G(T^{m_k+1}x, T^{m_k+1}x, T^{n_k+1}x)) = \psi(G(TT^{m_k}x, TT^{m_k}x, TT^{n_k}x))$$

$$\leq \psi((M(T^{m_k}x, T^{m_k}x, T^{n_k}x))) - \varphi((M(T^{m_k}x, T^{m_k}x, T^{n_k}x)))$$

$$\leq \psi((M(T^{m_k}x, T^{m_k}x, T^{n_k}x))).$$

making $k \to \infty$,

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu) \leq \psi(\mu),$$

is a contradiction unless $\mu = 0$.

But if $\mu = 0$ then it contradicts our assumption that $\mu > 0$, therefore $\{T^n x\}$ is $G$-Cauchy. Since $X$ is $G$-complete there exists a point $z \in \bigcap_{i=1}^{p} A_i$ such that $\{T^n x\}$ converges to $z$. Now for some $i \in \{1, 2, ..., p\}$ there exists sequences $\{T^{2n}x\}$ and $\{T^{2n-1}x\}$ in $A_i$ and $A_{i+1}$ respectively, with $A_{p+1} = A_1$, both converging to $z$.

Using (4.4.10), we get

$$\psi(G(T^{2n}x, T^{2n}x, Tz)) = \psi(G(TT^{2n-1}x, TT^{2n-1}x, Tz)) =$$

$$\leq \psi(M(T^{2n-1}x, T^{2n-1}x, z)) - \varphi(M(T^{2n-1}x, T^{2n-1}x, z)).$$

$$\leq \psi(M(T^{2n-1}x, T^{2n-1}x, z)).$$

Making $n \to \infty$, we get

$$\psi(G(z, z, Tz)) \leq \psi(G(z, z, z)) = \psi(0) = 0,$$

and $\psi(G(z, z, Tz)) = 0$. This implies $G(z, z, Tz) = 0$ and $z = Tz$.

Uniqueness of the fixed point is oblivious. □

## 4.5 Conclusion

In this chapter we have extended the existing fixed point theorems of complete metric space to complete $G$-metric space for the maps satisfying the following contractive conditions:

- Cyclic Contraction
- Weak $(\psi, \varphi)$-Contraction

# Chapter 5

# Application of Fixed Point Theory in solving Nonlinear Equations

The intent of this chapter is to study some renowned iterative methods and to do a comparative study for their convergence in 1D, 2D and 3D.

## 5.1 Introduction

Various generalizations of fixed point theorems in several other spaces such as probabilistic metric spaces, fuzzy metric spaces, uniform spaces, $b$-metric spaces etc have been studied. However in numerical praxis, the theory of approximate fixed points is more appropriate for actual applications and approximating a fixed-point solution (see [5], [37], [80], [126], [136] and [137]). Several researchers ([8], [21], [52], [155], [201], [202] etc.) have studied the convergence of the well known Ishikawa iteration, Jungck iteration, Mann iteration, Picard iteration etc. and drawn numerous results in their comparative analysis. Our work presents a stimulating result that is entirely different from the results which were declared previously in this field. We have undergone a comparative analysis among Picard, Mann, and Ishikawa iteration by using Matlab programming.

## 5.2 Method and Tools

When we use the same formula repeatedly to achieve and aim and that too using the result of the previous step in the following step that method is called an iterative method. Joseph Liouville [111] was the first to introduce the method of successive approximation in 1837 in connection with the study of linear differential equations of second order. It was later extended by J. Caque in 1864, L. Fuchs in 1870 and G. Peano in 1888 to the study of linear equations of order $n$.

To encompass nonlinear differential equations as well Charles Emile Picard [131] extended this method. This process is referred by mathematicians as the Picard iterations or function iterations. In linear and nonlinear analysis, computational analysis and several other areas of applied mathematics this method play an important role. Several extensions and generalizations of this method have been witnessed by the previous century.

An important role in fixed point theory to solve equations is played by iterative procedures. Speed plays a vital role in computations, it is of interest to know which iterative procedures converge faster to the desired solution.

The most prominent iterations that have maintained their magnificent presence in nonlinear analysis are as follows:

**(a)** Picard Iterations

**(b)** Mann Iterations

**(c)** Ishikawa iterations and

This chapter offers an experimental analysis of (a), (b) and (c) to find the roots of cubic equation.

### 5.2.1 Picard iterative procedure

Let $X$ be a space and $T$ a self-map of $X$. Then the Picard iteration can applied on $Tx = x$ with an initial choice $x_0$, and a sequence $\{x_n\}$ is generated by

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, ... \tag{5.2.1}$$

If the sequence $\{x_n\}$ converges to some point $z$ (say), then in many cases it turns out to be a solution.

### 5.2.2 Mann iterative procedure

It was first considered by W. R. Mann [114].

Let $A$ be a lower triangular matrix with nonnegative entries. Define

$$z_{n+1} = T(v_n), \text{ where } v_n = \sum a_{nk} z_k.$$

Let $\{\alpha_n\}$ be a sequence of nonnegative numbers such that

**(M1)** $\alpha_0 = 1$,

**(M2)** $0 \leq \alpha_n < 1$ for $n > 0$, and

**(M3)** $\sum \alpha_n = \infty$.

Then the entries of $A$ become $a_{nn} = \alpha_n$, $a_{n_k} = \alpha_k \prod(1 - \alpha_j), k < n$.

The above representation for $A$ leads to the following form:

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n T z_n \tag{5.2.2}$$

Notice that for $\alpha_n = 1$, this reduces to the Picard iterative process. Mann in [114] showed that, if $T$ is a continuous self-map of a closed interval $[a, b]$ with at most one fixed point, then his iteration scheme with $\alpha_n = \frac{1}{(n+1)}$ converges to the fixed point of $T$.

### 5.2.3 Ishikawa iterative procedure

If $E$ is a convex compact subset of a Hilbert space $H$, $T$ is a Lipschitzian pseudo-contractive map from $E$ into itself and $x_1$ is any point in $E$, then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$, where $x_n$ is defined iteratively for each positive integer $n$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] \tag{5.2.3}$$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers that satisfy the following three conditions:

**(I1)** $0 \leq \alpha_n \leq \beta_n \leq 1$

**(I2)** $\lim n \to \infty \beta_n = 0$

**(I3)** $\sum \alpha_n \beta_b = \infty$.

Let $\alpha_n = s$ and $\beta_n = s'$ then the Ishikawa iteration can be written as

$$\begin{aligned} y_n &= (1 - s')x_n + s'Tx_n \\ x_n &= (1 - s)x_n + sTy_n, n \geq 0 \end{aligned}$$

Iteration is the repetition of a process integrally over and over again. We begin with a seed for the iteration to iterate a function. This is a (real or complex) number $x_0$. Applying the function to $x_0$ yields the new number, $x_1$, say. Usually the iteration proceeds using the result of the previous computation as the input for the next. A sequence of numbers $x_0$, $x_1$, $x_2$, is then generated.

## 5.3 Comparative study of iteration methods

Here for our comparative analysis we have taken $s, s' \in (0, 1)$ and have derived fixed points iteration for the solution of cubic polynomial. In this section we have studied the comparisons on various iterative methods to find out the roots of following cubic equation

$$az^3 + bz^2 + cz + d = 0 \tag{5.3.1}$$

which can be represented as a functions $f(z) = \left[-\frac{1}{a}(bz^2 + cz + d)\right]^{\frac{1}{3}}$.

Let the initial values are $z = 0$, $a = 9$, $b = 4$, $c = -7$ and $d = -3$.

Table 5.1: Picard iteration for cubic polynomial

| $n$ | $f(z)$ |
|---|---|
| 1 | 0.70000 |
| 2 | 0.87066 |
| 3 | 0.87660 |
| 4 | 0.87660 |
| 5 | 0.87660 |
| 6 | 0.87660 |
| 7 | 0.87660 |
| 8 | 0.87660 |
| 9 | 0.87660 |
| 10 | 0.87660 |

Here we observe that the value of $f(z)$ converges to a fixed point after $3^{rd}$ iterations.

Table 5.2: Mann Iteration for cubic polynomial

| $n$ | $f(z)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $s = 0.3$ | $s = 0.4$ | $s = 0.5$ | $s = 0.6$ | $s = 0.7$ | $s = 0.8$ | $s = 0.9$ |
| 1 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 2 | 0.20801 | 0.27734 | 0.34668 | 0.41602 | 0.48535 | 0.55469 | 0.62403 |
| 3 | 0.37983 | 0.48700 | 0.58289 | 0.66677 | 0.73801 | 0.79603 | 0.84026 |
| 4 | 0.51383 | 0.63083 | 0.72136 | 0.78761 | 0.83248 | 0.85952 | 0.87276 |
| 5 | 0.61489 | 0.72448 | 0.79670 | 0.84012 | 0.86312 | 0.87315 | 0.87622 |
| 6 | 0.68943 | 0.78357 | 0.83606 | 0.86187 | 0.87254 | 0.87591 | 0.87656 |
| 7 | 0.74358 | 0.82014 | 0.85618 | 0.87069 | 0.87538 | 0.87646 | 0.87660 |
| 8 | 0.78248 | 0.84249 | 0.86636 | 0.87424 | 0.87624 | 0.87658 | 0.87660 |
| 9 | 0.81022 | 0.85606 | 0.87147 | 0.87566 | 0.87649 | 0.87660 | 0.87660 |
| 10 | 0.82989 | 0.86425 | 0.87404 | 0.87622 | 0.87657 | 0.87660 | 0.87660 |
| 11 | 0.84379 | 0.86918 | 0.87532 | 0.87645 | 0.87659 | 0.87660 | 0.87660 |
| 12 | 0.85358 | 0.87215 | 0.87596 | 0.87654 | 0.87660 | 0.87660 | 0.87660 |
| 13 | 0.86046 | 0.87393 | 0.87628 | 0.87658 | 0.87660 | 0.87660 | 0.87660 |
| 14 | 0.86529 | 0.87500 | 0.87644 | 0.87659 | 0.87660 | 0.87660 | 0.87660 |
| 15 | 0.86868 | 0.87564 | 0.87652 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 16 | 0.87105 | 0.87603 | 0.87656 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 17 | 0.87272 | 0.87626 | 0.87658 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 18 | 0.87388 | 0.87640 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 19 | 0.87470 | 0.87648 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 20 | 0.87527 | 0.87653 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 21 | 0.87567 | 0.87656 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 22 | 0.87595 | 0.87658 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 23 | 0.87615 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 24 | 0.87628 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |

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**Table 5.2 – Mann Iteration for Cubic polynomial**

| $n$ | $f(z)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $s = 0.3$ | $s = 0.4$ | $s = 0.5$ | $s = 0.6$ | $s = 0.7$ | $s = 0.8$ | $s = 0.9$ |
| 25 | 0.87638 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 26 | 0.87645 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 27 | 0.87649 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 28 | 0.87653 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 29 | 0.87655 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 30 | 0.87657 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 31 | 0.87658 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 32 | 0.87658 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 33 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 34 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 35 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 36 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 37 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 38 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 39 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 40 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |

Here we observe that the value of $f(z)$ converges to the fixed point after very few iterations if the value of $s$ is near to 1.

Table 5.3: Ishikawa iteration for cubic polynomial

| $n$ | $f(z)$ $at$ $s = 0.6$ $and$ $s' < 1$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $s' = 0.3$ | $s' = 0.4$ | $s' = 0.5$ | $s' = 0.6$ | $s' = 0.7$ | $s' = 0.8$ | $s' = 0.9$ |
| 1 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 2 | 0.55164 | 0.59183 | 0.63013 | 0.66677 | 0.70192 | 0.73569 | 0.76818 |
| 3 | 0.77683 | 0.80215 | 0.82308 | 0.84012 | 0.85366 | 0.86403 | 0.87146 |
| 4 | 0.84790 | 0.85835 | 0.86572 | 0.87069 | 0.87382 | 0.87559 | 0.87640 |
| 5 | 0.86851 | 0.87220 | 0.87442 | 0.87566 | 0.87627 | 0.87652 | 0.87659 |
| 6 | 0.87433 | 0.87555 | 0.87617 | 0.87645 | 0.87656 | 0.87660 | 0.87660 |
| 7 | 0.87597 | 0.87635 | 0.87652 | 0.87658 | 0.87660 | 0.87660 | 0.87660 |
| 8 | 0.87643 | 0.87654 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 9 | 0.87655 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 10 | 0.87659 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 11 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 12 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 13 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 14 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 15 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 16 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 17 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 18 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 19 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |
| 20 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 | 0.87660 |

Here we observe that the value of $f(z)$ converges to a fixed point after $10^{th}$ iterations, also it takes less number of iteration for $s'$ near to 1 at $s = 0.6$.

## 5.4 Fixed Point Iteration in 2D and 3D

Iterative techniques will now be introduced that extend the fixed point for finding a root of the system of nonlinear equations. We desire to have a method for finding a solution for the system of nonlinear equations.

$$\begin{aligned} f_1(x, y) &= 0 \\ f_2(x, y) &= 0 \end{aligned} \tag{5.4.1}$$

and

$$\begin{aligned} f_1(x, y, z) &= 0 \\ f_2(x, y, z) &= 0 \\ f_3(x, y, z) &= 0 \end{aligned} \tag{5.4.2}$$

Each equation in (5.4.1) and (5.4.2) implicitly defines a curve in the plane and we want to find their points of intersection as fixed point.

**Definition 5.4.1.** [59] A fixed point in $2D$ for the system of two equations

$$\begin{aligned} x &= f_1(x, y) \\ y &= f_2(x, y) \end{aligned} \tag{5.4.3}$$

is a point $(a, b)$ such that $a = f_1(a, b)$ and $b = f_2(a, b)$.

**Definition 5.4.2.** [59] A fixed point in $3D$ for the system of equations

$$\begin{aligned} x &= f_1(x, y, z) \\ y &= f_2(x, y, z) \\ z &= f_3(x, y, z) \end{aligned} \tag{5.4.4}$$

is a point $(a, b, c)$ such that $a = f_1(a, b, c)$, $b = f_2(a, b, c)$. and $c = f_3(a, b, c)$.

**Definition 5.4.3.** [59] A fixed point iteration in $2D$ for the system of two equations can be defined as

$$\begin{aligned} x_{k+1} &= f_1(x_k, y_k) \\ y_{k+1} &= f_2(x_k, y_k), \text{for } k = 0, 1, 2... \end{aligned} \tag{5.4.5}$$

Such that $(x_k, y_k)$ will be the fixed point for the (5.4.1), similarly $(x_k, y_k, z_k)$ will be the fixed point in $3D$ for (5.4.2), if

$$\begin{aligned} x_{k+1} &= f_1(x_k, y_k, z_k) \\ y_{k+1} &= f_2(x_k, y_k, z_k) \\ z_{k+1} &= f_3(x_k, y_k, z_k), \text{ for } k = 0, 1, 2... \end{aligned} \tag{5.4.6}$$

**Theorem 5.4.1.** *[59] Assume that all the functions and their first partial derivatives are continuous on a region that contains the fixed point $(a, b)$ or $(a, b, c)$ respectively. If the starting point for iteration is chosen sufficiently close to the fixed point, then one of the following cases apply.*

**Case 1:** *If* $(x_0, y_0)$ *is sufficiently close to* $(a, b)$ *and if*

$$\left|\frac{\partial f_1}{\partial x}\right|_{(a,b)} + \left|\frac{\partial f_1}{\partial y}\right|_{(a,b)} < m,$$
$$\left|\frac{\partial f_2}{\partial x}\right|_{(a,b)} + \left|\frac{\partial f_2}{\partial y}\right|_{(a,b)} < m,$$

*for* $0 < m < 1$, *then fixed point iteration will converge to the fixed point* $(a, b)$.

**Case 2:** *If* $(x_0, y_0, z_0)$ *is sufficiently close to* $(a, b, c)$ *and if*

$$\left|\frac{\partial f_1}{\partial x}\right|_{(a,b,c)} + \left|\frac{\partial f_1}{\partial y}\right|_{(a,b,c)} + \left|\frac{\partial f_1}{\partial z}\right|_{(a,b,c)} < m$$
$$\left|\frac{\partial f_2}{\partial x}\right|_{(a,b,c)} + \left|\frac{\partial f_2}{\partial y}\right|_{(a,b,c)} + \left|\frac{\partial f_2}{\partial z}\right|_{(a,b,c)} < m$$
$$\left|\frac{\partial f_3}{\partial x}\right|_{(a,b,c)} + \left|\frac{\partial f_3}{\partial y}\right|_{(a,b,c)} + \left|\frac{\partial f_3}{\partial z}\right|_{(a,b,c)} < m$$

*for* $0 < m < 1$, *then fixed point iteration will converge to the fixed point* $(a, b, c)$.

*If these conditions are not met, the iteration might diverge, which is usually the case.*

### 5.4.1 Algorithm for Nonlinear Systems in $2D$

In two dimensions we solve the nonlinear equation (5.4.1) for fixed point system as (5.4.3) and provide one initial approximation $\vec{x}_0 = (x_0, y_0)$ and generate a sequence $\{\vec{x}_k\} = \{(x_k, y_k)\}$ which converges to the solution $(a, b)$.

### 5.4.2 Algorithm for Nonlinear Systems in $3D$

In three dimensions, we solve the nonlinear equation (5.4.2) for fixed point system as (5.4.4) and provide one initial approximation $\vec{x}_0 = (x_0, y_0, z_0)$ and generate a sequence $\{\vec{x}_k\} = \{(x_k, y_k, z_k)\}$ which converges to the solution $(a, b, c)$.

**Example 5.4.1.** Solve following system of nonlinear equations

$$\begin{aligned} a_1x^2 + b_1y^2 + c_1x + d_1y + e_1 = 0 \\ a_2x^2 + b_2y^2 + c_2x + d_2y + e_2 = 0 \end{aligned} \tag{5.4.7}$$

by using fixed point iteration method.

To solve (5.4.7) we'll consider following two cases:

**Case 1:** As a first case we choose the constants $a_1 = -5$, $b_1 = -2$, $c_1 = 0$, $d_1 = 0$, $e_1 = 9$, $a_2 = 1$, $b_2 = -1$, $c_2 = -4$, $d_2 = 0$ and $e_2 = 5$ in (5.4.7). After replacing the constants,

$$\begin{aligned} -5x^2 - 2y^2 + 9 = 0 \\ x^2 - y^2 - 4x + 5 = 0 \end{aligned} \tag{5.4.8}$$

we have converted the above equation as (5.4.5) and found

$$x_{k+1} = f_1(x_k, y_k) = \frac{9x_k - 5x_k^2 - 2y_k^2 + 9}{9}$$

$$y_{k+1} = f_2(x_k, y_k) = \frac{-4x_k + x_k^2 + 11y_k - y_k^2 + 5}{11}$$

Now to generate the sequence of iterates we use above algorithm 5.5.1 and to initiate the iteration let us chose $x_0 = 1$ and $y_0 = 1.5$. The sequence of iterates and the values of $x_{k+1}$ and $y_{k+1}$ are available in the following table.

Table 5.4: Fixed point iteration in $2D$ for Case 1

| $n$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | 1.5 |
| 1 | 0.94444 | 1.47727 |
| 2 | 0.96394 | 1.47108 |
| 3 | 0.96682 | 1.46284 |
| 4 | 0.97199 | 1.45625 |
| 5 | 0.97586 | 1.45045 |
| 6 | 0.97929 | 1.44545 |
| 7 | 0.98221 | 1.44114 |
| 8 | 0.98472 | 1.43741 |
| 9 | 0.98687 | 1.43420 |
| 10 | 0.98871 | 1.43143 |
| 11 | 0.99030 | 1.42904 |
| 12 | 0.99166 | 1.42698 |
| 13 | 0.99283 | 1.42520 |
| 14 | 0.99383 | 1.42367 |
| 15 | 0.99470 | 1.42236 |
| 16 | 0.99544 | 1.42122 |
| 17 | 0.99608 | 1.42025 |
| 18 | 0.99663 | 1.41941 |
| 19 | 0.99710 | 1.41868 |

Continued on next page

**Table 5.4 – Fixed point iteration in $2D$**

| $n$ | $x_k$ | $y_k$ |
|---|---|---|
| 20 | 0.99750 | 1.41806 |
| 21 | 0.99785 | 1.41753 |
| 22 | 0.99815 | 1.41706 |
| 23 | 0.99841 | 1.41667 |
| 24 | 0.99863 | 1.41632 |
| 25 | 0.99882 | 1.41603 |
| 26 | 0.99899 | 1.41578 |
| 27 | 0.99913 | 1.41556 |
| 28 | 0.99925 | 1.41537 |
| 29 | 0.99936 | 1.41521 |
| 30 | 0.99945 | 1.41507 |
| 31 | 0.99952 | 1.41495 |
| 32 | 0.99959 | 1.41485 |
| 33 | 0.99965 | 1.41476 |
| 34 | 0.99970 | 1.41468 |
| 35 | 0.99974 | 1.41462 |
| 36 | 0.99977 | 1.41456 |
| 37 | 0.99981 | 1.41451 |
| 38 | 0.99983 | 1.41447 |
| 39 | 0.99986 | 1.41444 |
| 40 | 0.99988 | 1.41440 |
| 41 | 0.99989 | 1.41438 |
| 42 | 0.99991 | 1.41435 |
| 43 | 0.99992 | 1.41434 |
| 44 | 0.99993 | 1.41432 |

Continued on next page

**Table 5.4 – Fixed point iteration in $2D$**

| $n$ | $x_k$ | $y_k$ |
|---|---|---|
| 45 | 0.99994 | 1.41430 |
| 46 | 0.99995 | 1.41429 |
| 47 | 0.99996 | 1.41428 |
| 48 | 0.99996 | 1.41427 |
| 49 | 0.99997 | 1.41426 |
| 50 | 0.99997 | 1.41426 |
| 51 | 0.99998 | 1.41425 |
| 52 | 0.99998 | 1.41425 |
| 53 | 0.99998 | 1.41424 |
| 54 | 0.99998 | 1.41424 |
| 55 | 0.99999 | 1.41423 |
| 56 | 0.99999 | 1.41423 |
| 57 | 0.99999 | 1.41423 |
| 58 | 0.99999 | 1.41423 |
| 59 | 0.99999 | 1.41422 |
| 60 | 0.99999 | 1.41422 |
| 61 | 0.99999 | 1.41422 |
| 62 | 1.00000 | 1.41422 |
| 63 | 1.00000 | 1.41422 |
| 64 | 1.00000 | 1.41422 |
| 65 | 1.00000 | 1.41422 |

Here we observe that the value of $(x_k, y_k)$ converges to a fixed point after $60^{th}$ iterations.

**Case 2:** To consider second case on system of nonlinear equation (5.4.7), we are now choosing the constants $a_1 = -4$, $b_1 = 1$, $c_1 = 0$, $d_1 = 0$, $e_1 = 1$, $a_2 = -1$, $b_2 = -1$, $c_2 = 2$, $d_2 = 0$ and $e_2 = 3$. After replacing the constants,

$$\begin{aligned} -4x^2 + y^2 + 1 = 0 \\ -x^2 - y^2 + 2x + 3 = 0 \end{aligned} \tag{5.4.9}$$

we have converted the above equation as (5.4.5) and found

$$x_{k+1} = f_1(x_k, y_k) = \frac{8x_k - 4x_k^2 + y_k^2 + 1}{8}$$

$$y_{k+1} = f_2(x_k, y_k) = \frac{2x_k - x_k^2 + 4y_k - y_k^2 + 3}{4}$$

Now to generate the sequence of iterates we use above algorithm 5.5.1 and to start the iteration let us chose $x_0 = 1$ and $y_0 = 2$. The sequence of iterates and the values of $x_{k+1}$ and $y_{k+1}$ are available in the following table.

Table 5.5: Fixed point iteration in $2D$ for Case 2

| $n$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 1.125 | 2 |
| 2 | 1.117188 | 1.996094 |
| 3 | 1.116182 | 1.996563 |
| 4 | 1.116534 | 1.996622 |
| 5 | 1.116523 | 1.996602 |
| 6 | 1.116514 | 1.996603 |

Continued on next page

**Table 5.5 – Fixed point iteration in $2D$**

| $n$ | $x_k$ | $y_k$ |
|---|---|---|
| 7 | 1.116515 | 1.996603 |
| 8 | 1.116515 | 1.996603 |
| 9 | 1.116515 | 1.996603 |
| 10 | 1.116515 | 1.996603 |
| 11 | 1.116515 | 1.996603 |
| 12 | 1.116515 | 1.996603 |
| 13 | 1.116515 | 1.996603 |
| 14 | 1.116515 | 1.996603 |
| 15 | 1.116515 | 1.996603 |
| 16 | 1.116515 | 1.996603 |
| 17 | 1.116515 | 1.996603 |
| 18 | 1.116515 | 1.996603 |
| 19 | 1.116515 | 1.996603 |
| 20 | 1.116515 | 1.996603 |

Here we observe that the value of $(x_k, y_k)$ converges to a fixed point after $6^{th}$ iterations only.

**Example 5.4.2.** Solve the following simultaneous equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \\ a_3x + b_3y + c_3z + d_3 &= 0 \end{aligned} \tag{5.4.10}$$

by using fixed point iteration method.

To solve (5.4.10) we will consider following two cases:

**Case 1:** Let us choose $a_1 = 6$, $b_1 = 1$, $c_1 = 1$, $d_1 = -105$, $a_2 = 4$, $b_2 = 8$, $c_2 = 3$, $d_2 = -155$, $a_3 = 5$, $b_3 = 4$, $c_3 = -10$, and $d_3 = -65$ in equation (5.4.10). After replacing the constants, we have

$$
\begin{aligned}
6x + y + z - 105 &= 0 \\
4x + 8y + 3z - 155 &= 0 \\
5x + 4y - 10z - 65 &= 0
\end{aligned}
\tag{5.4.11}
$$

now to iterate with fixed point method convert the equation as following:

$$
\begin{aligned}
x_{k+1} &= \frac{105 - 0.x_k - y_k - z_k}{6} \\
y_{k+1} &= \frac{155 - 4x_k - 0y_k - 3z_k}{8} \\
z_{k+1} &= \frac{65 - 5x_k - 4y_k - 0z_k}{-10}
\end{aligned}
$$

Now to generate the sequence of iterates we use above algorithm 5.5.1 and to start the iteration let us chose $x_0 = 11$, $y_0 = 12$ and $y_0 = 13$. The sequence of iterates and the values of $x_{k+1}$, $y_{k+1}$ and $z_{k+1}$ are available in the following table.

Table 5.6: Fixed point iteration in $3D$ for Case 1

| $n$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 0 | 11 | 12 | 13 |
| 1 | 13.33333 | 9.00000 | 3.80000 |
| 2 | 15.36667 | 11.28333 | 3.76667 |
| 3 | 14.99167 | 10.27917 | 5.69667 |
| 4 | 14.83736 | 9.74292 | 5.10750 |
| 5 | 15.02493 | 10.04101 | 4.81585 |
| 6 | 15.02386 | 10.05659 | 5.02887 |
| 7 | 14.98576 | 9.97725 | 5.03457 |
| 8 | 14.99803 | 9.99416 | 4.98378 |
| 9 | 15.00368 | 10.00707 | 4.99668 |
| 10 | 14.99938 | 9.99941 | 5.00467 |
| 11 | 14.99932 | 9.99856 | 4.99945 |
| 12 | 15.00033 | 10.00055 | 4.99909 |
| 13 | 15.00006 | 10.00018 | 5.00038 |
| 14 | 14.99991 | 9.99983 | 5.00010 |
| 15 | 15.00001 | 10.00001 | 4.99988 |
| 16 | 15.00002 | 10.00004 | 5.00001 |
| 17 | 14.99999 | 9.99999 | 5.00002 |
| 18 | 15.00000 | 9.99999 | 4.99999 |
| 19 | 15.00000 | 10.00000 | 5.00000 |
| 20 | 15.00000 | 10.00000 | 5.00000 |
| 21 | 15.00000 | 10.00000 | 5.00000 |
| 22 | 15.00000 | 10.00000 | 5.00000 |
| 23 | 15.00000 | 10.00000 | 5.00000 |
| 24 | 15.00000 | 10.00000 | 5.00000 |

Continued on next page

**Table 5.6 – Fixed point iteration in $3D$**

| $n$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 25 | 15.00000 | 10.00000 | 5.00000 |

Here we observe that the value of $(x_k, y_k, z_k)$ converges to a fixed point after $17^{th}$ iterations only.

**Case 2:** Now taking $a_1 = 10$, $b_1 = -2$, $c_1 = -3$, $d_1 = -205$, $a_2 = 2$, $b_2 = -10$, $c_2 = 2$, $d_2 = 154$, $a_3 = 2$, $b_3 = 1$, $c_3 = -10$, and $d_3 = 120$ in equation (5.4.10) as second case. After replacing the constants, we have

$$10x - 2y - 3z - 205 = 0$$

$$2x - 10y + 2z + 154 = 0 \tag{5.4.12}$$

$$2x + y - 10z + 120 = 0$$

now to iterate with fixed point method converting the equation as following:

$$x_{k+1} = \frac{0.x_k + 2y_k + 3z_k + 205}{10}$$

$$y_{k+1} = \frac{2x_k + 0y_k + 2z_k + 154}{10}$$

$$z_{k+1} = \frac{2x_k + y_k + 0z_k + 120}{10}$$

Now to generate the sequence of iterates we use above algorithm 5.5.1 and to start the iteration let us chose $x_0 = 1$, $y_0 = 2$ and $y_0 = 3$. The sequence of iterates and the values of $x_{k+1}$, $y_{k+1}$ and $z_{k+1}$ are available in the following table.

Table 5.7: Fixed point iteration in $3D$ for Case 2

| $n$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 21.80000 | 16.20000 | 12.40000 |
| 2 | 27.46000 | 22.24000 | 17.98000 |
| 3 | 30.34200 | 24.48800 | 19.71600 |
| 4 | 31.31240 | 25.41160 | 20.51720 |
| 5 | 31.73748 | 25.76592 | 20.80364 |
| 6 | 31.89428 | 25.90822 | 20.92409 |
| 7 | 31.95887 | 25.96367 | 20.96968 |
| 8 | 31.98364 | 25.98571 | 20.98814 |
| 9 | 31.99358 | 25.99436 | 20.99530 |
| 10 | 31.99746 | 25.99778 | 20.99815 |
| 11 | 31.99900 | 25.99912 | 20.99927 |
| 12 | 31.99961 | 25.99965 | 20.99971 |
| 13 | 31.99984 | 25.99986 | 20.99989 |
| 14 | 31.99994 | 25.99995 | 20.99996 |
| 15 | 31.99998 | 25.99998 | 20.99998 |
| 16 | 31.99999 | 25.99999 | 20.99999 |
| 17 | 32.00000 | 26.00000 | 21.00000 |
| 18 | 32.00000 | 26.00000 | 21.00000 |
| 19 | 32.00000 | 26.00000 | 21.00000 |
| 20 | 32.00000 | 26.00000 | 21.00000 |
| 21 | 32.00000 | 26.00000 | 21.00000 |
| 22 | 32.00000 | 26.00000 | 21.00000 |

Continued on next page

**Table 5.7 – Fixed point iteration in $3D$**

| $n$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 23 | 32.00000 | 26.00000 | 21.00000 |
| 24 | 32.00000 | 26.00000 | 21.00000 |
| 25 | 32.00000 | 26.00000 | 21.00000 |

Here we observe that the value of $(x_k, y_k, z_k)$ converges to a fixed point after $16^{th}$ iterations only.

## 5.5 Conclusions

A comparative analysis among Picard, Mann and Ishikawa iterations has been done. It was found by using Matlab programming, that Picard's iteration converges faster than the rest, followed by Ishikawa's iteration while Mann iteration converges slowly. Besides this, it was also observed that as if the value of $s$ and $s'$ increases, the convergence goes on faster for all Maan and Ishikawa iteration. The fixed point iteration method for solution of system of nonlinear equations in $2D$ and solution of simultaneous linear equations in $3D$ has also been studied. It was proved that the convergence is faster near to fixed point of the system of equations.

# Chapter 6

# Approximate Fixed Point Theorem

Fixed point theory has solutions to various problems in applied mathematics. Still, it has been proved by practice that in many real situations an approximate solution is more than enough. So there is no strict requirement of the existence of fixed point, but that of nearly fixed points. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve. Let $T$ be a self map of a metric space $(X, d)$. Let us look for an approximate solution of the equation $Tx = x$. If there exists a point $z \in X$ such that $d(Tz, z) \leq \varepsilon$ , where $\varepsilon$ is a positive number, then $z$ is called an approximate solution of the equation $Tx = x$, or equivalently, $z \in X$ is an approximate fixed point (or $\varepsilon$-fixed point) of $T$.

## 6.1 Approximate Fixed Point

The theory of fixed point and consequently of approximate fixed point finds application in mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equations and several other areas of applicable analysis (see, for instance, [28], [58], [64], [172], [184] and several references thereof).

Approximate fixed points by generalizing Brouwer fixed point theorem to a discontinuous map have been found by Cromme and Diener [47]. Hou and Chen [76] have extended their results to set valued maps. Interesting results in product spaces have been achieved by Espinola and Kirk [64]. Approximate fixed point theorems for contractive and non-expansive maps by weakening the conditions on the spaces have been discussed by Tijs et. al. [184]. R. Branzei et al [28] further extended these results to multifunctions in Banach spaces. Recently M. Berinde [18] obtained approximate fixed point theorems for operators satisfying Kannan, Chatterjea and Zamfirescu type of conditions on metric spaces.

The main intent of this chapter is to establish some approximate fixed point results in metric spaces under various contractive conditions.

Following definition are essentially due to Tijs et. al. [184].

**Definition 6.1.1.** Let $(X, d)$ be a metric space and $T : X \to X$. Let $\varepsilon$ be a positive number. Then a point $z \in X$ is an $\varepsilon$-fixed point of $T$ if $d(Tz, z) \leq \varepsilon$. A map $T : X \to X$ is said to have approximate fixed point property if, for each $\varepsilon > 0$ , the map $T$ possesses at least one $\varepsilon$-fixed point.

**Definition 6.1.2.** The set of all $\varepsilon$-fixed points of $T$ for a given $\varepsilon$, is defined as below

$$F_\varepsilon(T) = \{z \in X : \ z \text{ is an } \varepsilon - \text{fixed point of } \ T\}.$$

**Definition 6.1.3.** Let $T : X \to X$, then $T$ has the **approximate fixed point property** if

$$\forall \varepsilon > 0, F_\varepsilon(T) \neq \phi$$

**Definition 6.1.4.** Let $(X, d)$ be a metric space, $T, S : X \to X$ then $S$ is said to be $T$-asymptotic regular if,

$$d(TS^n(z), TS^{n+1}(z)) \to 0 \ \text{ as } \ n \to \infty, \ \ \forall z \in X$$

**Definition 6.1.5.** A self map $T : X \to X$ on metric space $(X, d)$ is said to be subsequentially convergent if we have, for every sequence $\{y_n\} \in X$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

In order to prove our results we need following lemma [139].

**Lemma 6.1.1.** Let $(X, d)$ be a metric space and $T, S : X \to X$ be two commuting maps. If $S$ is $T$-asymptotically regular, then $S$ has an approximate fixed point property.

## 6.2 Approximate Fixed Point Theorem

In this chapter we have obtained following approximate fixed point theorem:

**Theorem 6.2.1.** *Let $(X, d)$ be a complete metric space and $T, S : X \to X$ be mappings such that $T$ is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \leq \lambda d(Tx, TSx) + d(Ty, TSy), \forall x, y \in X, \tag{6.2.1}$$

*then for $\varepsilon > 0$ $F_\varepsilon(S) \neq \phi$,*

*i.e. $S$ has approximate fixed point property.*

*Proof.* Let $x_0$ be any arbitrary point in $X$. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, ....$ Now using the inequality (6.2.1), we have

$$d(Tx_n, Tx_{n+1}) = d(TSx_{n-1}, TSx_n)$$
$$\leq \lambda \left[ d(Tx_{n-1}, TSxn-1) + d(Tx_n, TSx_n) \right] \tag{6.2.2}$$

So

$$d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \tag{6.2.3}$$

By using the argument repeatedly,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \\ &\leq \left( \frac{\lambda}{1-\lambda} \right)^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leq ... \leq \left( \frac{\lambda}{1-\lambda} \right)^n d(Tx_0, Tx_1) \end{aligned} \tag{6.2.4}$$

i.e.

$$d(TS^n x_0, TS^{n+1} x_0) \leq \left( \frac{\lambda}{1-\lambda} \right)^n d(Tx_0, Tx_1) \tag{6.2.5}$$

Since $\lambda \in [0, \frac{1}{2})$, from the above inequality we get that $d(TS^n x_0, TS^{n+1} x_0) \to 0$ as $n \to \infty$, for all $x \in X$, which implies that $S$ is $T$-asymptotically regular. Now by applying lemma (6.1.1) we obtain that for every $\varepsilon > 0$,

$$F_\varepsilon(S) \neq \phi$$

which means that $S$ has approximate fixed point property. □

**Theorem 6.2.2.** *Let $X$ be a complete b-metric space with metric $d$ and let $T : X \to X$ be a function with the following property*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y) \tag{6.2.6}$$

$\forall x, y \in X$, *where $a, b, c$ are non-negative real numbers and satisfy $a + s(b + c) < 1$ for $s \geq 1$ then $T$ has an approximate fixed point property.*

*Proof.* Let $x_0 \in X$ and $\{x_n\}$ be a sequence in $X$, such that

$$x_n = Tx_{n-1} = T^n x_0 \tag{6.2.7}$$

Now

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq ad(x_n, Tx_n) + bd(x_{n-1}, Tx_{n-1}) + cd(x_n, x_{n-1}) \\ &= ad(x_n, x_{n+1}) + bd(x_{n-1}, x_n) + cd(x_n, x_{n-1}) \\ &\implies (1 - a)d(x_{n+1}, x_n) \leq (b + c)d(x_n, x_{n-1}) \\ &\implies d(x_{n+1}, x_n) \leq \frac{(b + c)}{(1 - a)} d(x_n, x_{n-1}) = pd(x_n, x_{n-1}) \end{aligned}$$

where $p = \frac{(b+c)}{(1-a)} < \frac{1}{s}$

Now using (6.2.7)

$$\implies d(T^{n+1}x_0, T^n x_0) \leq \frac{(b + c)}{(1 - a)} d(x_n, x_{n-1}) = pd(x_n, x_{n-1})$$

Continuing this process we can easily see that $d(T^{n+1}x_0, T^n x_0) \leq p^n d(x_0, x_1)$.

On taking $n \to \infty$ we get that $d(T^{n+1}x_0, T^n x_0) \to 0$, for all $x \in X$. Using the definition 2.3 [132] we can see $T$ is an asymptotically regular map. Now by applying Lemma (3.1) [132] on $T$ we obtain that for every $\varepsilon > 0$,

$$F_\varepsilon(T) \neq \phi$$

which means that $T$ has an approximate fixed point property. □

**Theorem 6.2.3.** *Let $(X, d)$ be a complete b-metric space with constant $s \geq 1$. Let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq k.d(x, y)$$

*with $k \in [0, 1)$ and $ks < 1$. Then $T$ has a approximate fixed point property.*

*Proof.* Let $x_0 \in X$ and there exist a sequence $\{x_n\} \in X$ such that

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, 3... \tag{6.2.8}$$

Since $T$ is a contraction with constant $k \in [0, 1)$, then we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq k.d(x_n, x_{n-1}) = k.d(Tx_{n-1}, Tx_{n-2}) \\ &\leq k^2.d(x_{n-1}, x_{n-2}) \leq ... \leq k^n.d(x_1, x_0) \end{aligned}$$

This implies $d(Tx_n, Tx_{n-1}) = d(T^{n+1}x_0, T^n x_0) \leq \frac{1}{s^n} d(x_1, x_0)$

Again if we take $n \to \infty$ we get that $d(T^{n+1}x_0, T^n x_0) \to 0$, for all $x \in X$. Using the definition 2.3 [132] we can see $T$ is an asymptotically regular map. Now by applying Lemma (3.1) [132] on $T$ we obtain that for every $\varepsilon > 0$,

$$F_\varepsilon(T) \neq \phi$$

which means that $T$ has an approximate fixed point property. $\square$

**Theorem 6.2.4.** *Let $(X, d)$ be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \mu.[d(x, Tx) + d(y, Ty)] \tag{6.2.9}$$

*for all $x, y \in X$ and $\mu \in \left[0, \frac{1}{2}\right]$ then $T$ has an approximate fixed point property.*

*Proof.* Let $x_0 \in X$ and there exist a sequence $\{x_n\} \in X$ such that

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, 3... \tag{6.2.10}$$

Now by using (6.2.10) and (6.2.9) we obtain,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \mu.[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leq \mu.[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

On simplifying we obtain

$$d(x_n, x_{n+1}) \leq \left(\frac{\mu}{1-\mu}\right) d(x_{n-1}, x_n) = \left(\frac{\mu}{1-\mu}\right)^n d(x_0, x_1) \tag{6.2.11}$$

Now,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1} x_1) \leq \left(\frac{\mu}{1-\mu}\right)^n d(x_0, x_1) \tag{6.2.12}$$

Since $\mu \in [0, \frac{1}{2})$, from the above inequality (6.2.12) we get that $d(T^n x_0, T^{n+1} x_0) \to 0$ as $n \to \infty$, for all $x \in X$, which implies that $T$ is asymptotically regular (as per the definition 2.3 [132]) Now by Lemma (3.1) [132] on $T$ we obtain that for every $\varepsilon > 0$,

$$F_\varepsilon(T) \neq \phi$$

which conclude that $T$ has approximate fixed point property. □

## 6.3 Conclusions

In this chapter we have gone through some existing approximate fixed point theorems and established some new fixed point theorems for complete metric space and for $b$-metric space for the maps satisfying following contractive conditions:

- $T$-Contraction
- Banach Contraction
- Kannan Contraction
- Reich Contraction

# Chapter 7

# Conclusions and Recommendations

## 7.1 Main Conclusion from the Present Study

"Fixed Point Theory" is a striking mixture of analysis as well as Topology. Vigorous research activities have been attracted by the study of fixed point theory. Fixed point theory is one of the most influential and rewarding tools of modern mathematics and may be deliberated as a core subject of nonlinear analysis. Excellent monographs and surveys by eminent authors about fixed point theory have appeared in the recent years.

An intrinsic property of a map is the presence or absence of a fixed point. However,

many essential or ample conditions for the existence of such points involve a mixture of algebraic, order theoretic or topological properties of the mapping or its domain.
Fixed point theorems give the conditions under which mapping have solutions. The theory of fixed points has been exposed as a very significant tools in the study of nonlinear phenomena over the last half century or so. Although basic ideas of fixed point theorems are found in the work of Augustin-Louis Cauchy (1789-1857) (see also Kirk and Sims [118]).

In 1906, M. Frechet introduced the concept of an abstract metric space. It furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a 'distance' appears. The aim of this work is to offer an integrated and comprehensive exposition of a metric fixed point theory and its various beneficial interactions with topological structures.

Metric fixed point theory is an indispensable part of mathematical analysis because of its applications in different areas like variational and linear inequalities, improvement and approximation theory. In developing the methods to solve the problems in mathematics and sciences the theory of fixed point in metric spaces plays a vital role.

The capability of solving numerous equations through metric fixed point is very high. To overcome the problem of measurable functions w.r.t. a measure and their convergence, Czerwik needs an extension of metric space. Using this idea, he presented a generalization of the renowned Banachs fixed point theorem in the $b$-metric spaces. Our aim is to demonstrate the validity of some important fixed point results into $b$-metric spaces. In this regard we have obtained received fixed point results in complete $b$-metric space which are the extensions of the theorems given by Reich and Hardy-Rogers.

Huang and Zhang [77] considered such spaces under the name of cone metric spaces, defined convergence and Cauchy sequence in term of interior points of the underlying cone. Huang, Zhang and other researchers proved some fixed point and common fixed point theorems for contractive-type mappings in cone metric spaces and cone uniform spaces. Some fixed point theorems in cone metric spaces have been derived for:

- $T$-contraction
- Compatible maps
- Asymptotically regular maps

Mustafa and Sims introduced a more appropriate generalization of metric spaces, that of $G$-metric spaces this was done to overcome fundamental flaws in Dhage's theory [54] of generalized metric spaces . Afterwards, Mustafa et. al. obtained several fixed point theorems for mappings satisfying different contractive conditions in $G$-metric spaces.

- Cyclic Contraction
- Weak $(\psi, \varphi)$ Contraction

The applicability of fixed point theorems in computer science has been enhanced by Tarski [181]. Fixed points are involved in program derivation which influence extensively the construction, reliability, maintenance and extensibility of a software this was found by Cai and Paige [33]. As an application of fixed point theory in solving nonlinear equations a comparative analysis among Picard, Mann and Ishikawa iterations has been done. It was found that Picard's iteration converges faster than the rest by using Matlab programming followed by Ishikawa's iteration while Mann iteration converges slowly. Besides this, it was also revealed that as if the value of $s$ and $s'$ increases, the convergence goes on faster for all Maan and Ishikawa iteration. We have also studied the fixed point iteration method for solution of system of nonlinear equations in $2D$ and solution of simultaneous linear equations in $3D$. We found that convergence is faster near to fixed point of the system of equations. The conclusion was that the fixed point iteration is the most convenient way to have the root of any equation in $1D$, $2D$ and $3D$.

Fixed point theory can solved multiple problems in applied mathematics. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of nearly fixed points. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve. We have studied some existing approximate fixed point theorems and developed some new fixed point theorems for complete metric space and for $b$-metric space for the maps satisfying following contractive conditions:

- $T$-Contraction
- Banach Contraction

- Kannan Contraction
- Reich Contraction

## 7.2 Recommendations based on present study

In the present study while deriving the fixed point theorems under various contractive conditions for $b$-metric space, cone metric space and $G$-metric space It was found that the several possibilities of study are there to establish some new fixed point results concerning approximate fixed points, coincidence point, endpoints, approximate endpoints and approximate best proximity points of the maps satisfying different contractive conditions.

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# List of Publications

[1] Mishra, P. K., Mohan, R., and Sachdeva S.,2013, Fixed point theorems for cyclic weak contraction in $G$-metric spaces, *Journal of Advanced Studies in Topology*, Vol. 4, Issue 2, pp. 18-25.

[2] Mishra P. K., Sachdeva S and Banerjee S.K., 2014, Some Fixed Point Theorems in $b$-Metric Space, *Turkish Journal of Analysis and Number Theory*, Vol. 2, Issue 1, pp. 19-21.

**Forthcoming Papers**

Mishra P. K., Sachdeva S and Banerjee S.K., 2014, Hundred Years of Brouwer Fixed Point Theorem, *Journal of Applied Mathematics and Informatics* (in review).