DESIGN AND MODELLING OF NONLINEAR DYNAMICAL SYSTEM

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THESIS COMPLETION CERTIFICATE

This is to certify that the thesis on “**Design and Modelling of Nonlinear Dynamical System”** by **Mukul Kumar Gupta** in Partial completion of the requirements for the award of the degree of Doctor of Philosophy (Engineering) is an original work carried out by him under our joint supervision and guidance. It is certified that the work has not been submitted anywhere else for the award of any other diploma or degree of this or any other university.

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ABSTRACT

The dynamic behavior of double and triple link pendula considering both the lumped and distributed mass has been studied. The system of governing differential equations for double pendulum (DP) and triple pendulum (TP) is derived using the Euler- Lagrange (EL) approach. Effect of damping of pivots is also considered in the model. The governing equations are also derived in the terms of momenta and angular velocity without considering the damping to confirm the results obtained by the E-L approach. Chaotic behavior of these systems was investigated using time series plot, Poincare map and Lyapunov exponent. It is observed that bottom pendulum is most sensitive than other pendula. Further, tendency of chaotic behavior increases with degree of freedom (dof) for the same initial conditions. Nature of Poincare maps of the TP becomes more intricate in comparison to the DP. Interestingly, damping of pivots results in regular motion of the pendulum system. In other words, introduction of damping in the multiple pendula reduces the tendency of chaos. Moreover, there is no qualitative difference between the lumped and distributed pendulum systems as far as the dynamical behavior is concerned. All the observations and results are validated experimentally. The LQR technique was used to control inverted triple pendulum. The control technique was found to be effective. This is also valid for double pendulum

EXECUTIVE SUMMARY

A pendulum is a highly nonlinear system used for validating control algorithms as well as dynamic model for variety of physical systems. In a pendulum either there can be free vibrations or forced vibration. The dynamic behavior of double and triple link pendula considering both the lumped and distributed mass has been studied under free vibration. The system of governing differential equations for Double Pendulum (DP) and Triple Pendulum (TP) is derived using the Euler-Lagrange (EL) approach. Effect of damping of pivots is also considered in the model. The governing equations are also derived in the terms of momenta and angular velocity without considering the damping to confirm the results obtained by the E-L approach. Chaotic behavior of these systems was investigated using time series plot, FFT Analysis, Poincare map and Lyapunov exponent.

It is observed that bottom pendulum is most sensitive than other pendula. Further, tendency of chaotic behavior increases with Degree of Freedom (DoF) for the same initial conditions. Also the effect of mass and length has been studied and it can be verified that in case of DP chaotic nature is mainly because of second mass and length. With the increment of lowest mass or lowest length Lyapunov exponent becomes positive which is a sign of chaos. Poincare becomes irregular in nature as we increase the lowest mass and length. Also it is verified for both DP and TP that as going from lumped to distributed system Poincare become more regular in nature and chaotic nature increases. Nature of Poincare

maps of the TP becomes more intricate in comparison to the DP. Interestingly, damping of pivots results in regular motion of the pendulum system. In other words, introduction of damping in the multiple pendula reduces the tendency of chaos.

The effect of frequency is also considered for SP, DP and TP and it can be found that that natural frequency of the linearized system increases from lumped to distributed system and also order of frequency will be like in case of top pendulum least one and Bottom Pendulum Highest one. Analysis of Linear stability is also being done for both Double and Triple Pendulum. Moreover, there is no qualitative difference between the lumped and distributed pendulum systems as far as the dynamical behavior is concerned. Also experimental analysis has been done for double and triple pendulum and it shows the chaotic behaviour. The LQR technique was used to control inverted triple pendulum. Using LQR the system stabilizes in less than 4 seconds. The control technique was found to be effective. This is also valid for double pendulum

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LIST OF FIGURES

LIST OF SYMBOLS

L	Lagrangian
H	Hamiltonian
p	Momenta
$\dot{p}$	Angular Momenta
Q_i	External Forces
T	Kinetic Energy
V	Potential Energy
ω_0	Fundamental Frequency
b	Friction Strength
f_d	Driving Force
ω_d	Driving Frequency
ω_n	Natural Frequency
I	Moment of Inertia
Q	Rayleigh Damping
J	Performance Index

LIST OF TABLES

ACRONYMS

DOF	Degree of Freedom
DP	Double Pendulum
DDP	Damped Driven Pendulum
FDDP	Forced damped driven pendulum
FFT	Fast Fourier Transform
LIGO	Laser Interferometer Gravitational Observatory
LQR	Linear Quadratic Regulator
LE	Lyapunov Exponent
KE	Kinetic Energy
MATLAB	Matrix Laboratory
ODE	Ordinary Differential Equation
PE	Potential Energy
SP	Simple Pendulum
TP	Triple Pendulum
TE	Total Energy
TLIP	Triple Link Inverted Pendulum

LIST OF ABBREVATIONS

m	Meter
cm	Centimeter
t	Time
f	Frequency
Hz	Hertz

CHAPTER 1

INTRODUCTION

The chapter describes the introduction to nonlinear systems and corresponding dynamic analysis and procedures. In the chapter an attempt is made to understand the dynamical behavior of nonlinear systems by obtaining differential equation based mathematical model. A brief insight into formation and evolution of differential equation and corresponding dynamical behavior is presented in this chapter. The objective of this research work is presented at the end of the chapter.

1.1 INTRODUCTION TO DYNAMICAL SYSTEM

A system is a combination of different interacting components or parts that is perceived as a single entity. An electric motor, an airplane and biological unit such as the human arm are examples of systems. A system is characterized by two properties, which are as follows;

1. The interrelations between the components that are contained within the system.
2. The system boundaries that separate the components within the system from the components outside.

The parts or components making up the system may be clearly or vaguely defined. These parts are connected through each other through a particular set of variables, called the states of the system, that completely determine the behaviorof the system at any given time .A dynamical system is a system whose state continuously changes with respect to time.

Specifically, the state of a dynamical system can be regarded as an information storage or memory of several past system events. Once the flow of a dynamical system describing the motion of the system starting from a given initial state is given, dynamical system theory can be used to describe the behavior of the system states over time for different initial conditions. The initial condition for the system should be well known for obtaining the dynamic trajectory i.e. identifying the behavior at any future point of time. Hence, the state of a dynamical system at a given time is uniquely determined by the state of the system at the initial time and the present input to the system. In other words, the state of a dynamical system in general depends on both the present input to the system and the past history of the system.

A dynamical system can be represented in terms of mathematical model by using set of differential equation in continuous time and discrete equation in discrete time consisting states, input and output of a given physical system. The key theoretical tool in the description of dynamics is a state space or phase space description of the behaviour of the system.

In dealing with a system, the effect of external quantities upon the behavior is focused. The external quantities act as the inputs to the system. The condition or the state of the system is described by the state variables. The state variable provides the information that, together with the knowledge of the system inputs, enables us to determine the future state of the system. "A dynamical system consists of a set of possible states, together with a rule that determines the present state in terms of past states".

On dynamical system modeled by a set of ordinary differential equations

$$x_i^{\cdot} = f_i(t, x_{1,} x_2, \dots \dots . x_{n,} u_1, u_2, \dots \dots \dots u_m), \quad (1.1)$$

$$x_i(t_0) = x_{i0,}\ i = 1,2, \dots . . n \quad (1.2)$$

together with p functions

$$y_j = h_j(t, x_1, x_2 \dots . x_n, u_1, u_2 \dots \dots . u_m), \qquad j = 1,2, \dots \dots . p \quad (1.3)$$

Where the system model state is $x = [x_1, x_2 \dots x_n]^T \in R^n$. The system input is $u = [u_1, u_2, \dots . . u_m]^T \in R^m$, and the system output is $y = [y_1, y_2, \dots \dots . y_p]^T \in R^P$.

In vector notation of the above system model has the form

$$\dot{x} = f(t, x(t), u(t)), \ \ x(t_0) = x_0 \quad (1.4)$$

$$y = h(t, x(t), u(t)), \quad (1.5)$$

Where $f: R \times R^n \times R^m \rightarrow R^n$ and $h: R \times R^n \times R^m \rightarrow R^p$ are vector valued functions [1].

System stability is characterized by analyzing the response of a dynamical system to small perturbations in the system states. Specifically, an equilibrium point of a dynamical system is said to be stable if, for sufficiently small values of initial disturbances, the perturbed motion remains in an arbitrarily prescribed small region of the state space. More precisely, stability is equivalent to continuity of solutions as a function of the system initial conditions over a neighborhood of the equilibrium point uniformly in time. If, in addition, all solutions of the dynamical system approach the equilibrium point for large values of time, then the equilibrium point is said to be asymptotically stable.

Since most of the physical systems are inherently nonlinear in nature and nonlinearity occur because of input constraints (saturation, dead band), Kinematic Effect, Gyroscopic Effects (rotational motion) and geometric constraints. However nonlinear system can exhibit a very rich dynamical behavior such as multiple equilibria, limit cycles, bifurcations, jump resonance phenomena and Chaos makes the study of nonlinear system difficult. Pendulum is a highly nonlinear system. The dynamics of simple pendulum, double pendulum and triple pendulum is analyzed for lumped and distributed mass both.

1.1.1 AUTONOMOUS AND NONAUTONOMOUS SYSTEM

The equation of motion were written as a system of equation

$$\dot{x} = f(x,u) \tag{1.6}$$

where $x = (q, \dot{q})$ is the state of the system. The parameters of the mechanism and the environment are denoted as u. These parameters include the link lengths, masses and the distance between walls. A system of ordinary differential equations is autonomous when it does not depend on time (or another independent variable). In contrast, a system is non-autonomous when it does depend on time. In Eqn. (1.6) since the right hand side does not include time it is an autonomous system. A non-autonomous system is of the form

$$\dot{x} = f(x,u,t); \qquad x,t \in \mathbb{R}^n \times \mathbb{R}, \tag{1.7}$$

A nth-order time-periodic non-autonomous system with period T i.e., $f(t) = f(t + T_0)$ can always be converted into an $(n+1)^{th}$ order autonomous system of differential equations.

1.2 TYPES OF DYNAMIC MOTIONS

There are three classic types of dynamic motions which are relevant to our climbing mechanism.

(1) *Equilibrium (fixed point)* - The notion of fixed points (also called stationary points or critical points or equilibrium points) in state space plays a key role in understanding the dynamics on nonlinear systems. If the system stat at one of these fixed point, it stays at that fixed point for all time. Since the time derivatives of the state space variable are zero at the fixed point, those variables cannot change in time [25]. For a one dimensional state space there are three fixed points.

(a) *Nodes (Sinks)* - Fixed points that attract nearby trajectories.

(b) *Repellers (Sources)* - Fixed points that repel nearby trajectories.

(c) *Saddle Points* - Fixed Points that attracts trajectories on one side will repel them on the other.

(2) *Periodic motion or a limit cycle* - With periodic motion the pendulum repeats its behaviour at regular intervals. In the simplest case, the period of the motion will coincide with the period of the forcing. The motion of the pendulum will either periodic or chaotic. In state space with two or more dimensions, it is possible to have cyclic or periodic behaviour. This very important kind of

behaviour is represented by closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called limit cycle if the cycle is isolated, that is if the trajectories nearby either approach or are repelled from limit cycle [25].

(3) *Quasi-periodicity and chaos* - The quasi periodicity involves competition, in general between two or more independent frequencies characterizing the dynamics of the system. This occurs in (at least) two kinds of system.

(i) A nonlinear system with a natural oscillation frequency, driven by an external periodic force.

(ii) Nonlinear systems that spontaneously develop oscillation at two frequencies as some parameter of the system are varied.

The quasi-periodic is used to describe the behaviour when the two frequencies are incommensurate (i.e. Frequency ratio is irrational) because, in fact, the system's behaviour never exactly repeats itself in that case [25]. If the two frequencies are incommensurate, the Poincare points will never (in principle) repeat. Indeed, the time behaviour of a quasi-periodic system can look quite irregular. The theory underlying the quasi-periodic route to chaos [54] tells us only that this scenario is likely to lead to chaotic behaviour; not that is must.

System shows the chaotic behaviour when moving from simple pendulum to multiple pendulums. Chaos is "aperiodic long term behavior in a deterministic system that exhibits sensitive dependence on initial conditions" [3]. Sensitive dependence to initial condition occurs when two very close initial conditions diverge exponentially from each other

1.3 TOOLS FOR THE MEASUREMENT OF CHAOS

For the measurement of chaotic behaviour following tools are used.

(i) *Time-series plot* – Time series either of the angular displacement or the angular velocity as a function of time is familiar geometrical devices and provides another picture of the motion. This is the simplest technique to see the evidence of chaos. In that case, two trajectories with very slight difference in initial value show considerable separation after a certain time in time-series plot.

(ii) *Phase-plot (Displacement-velocity plot)* - The motion of a pendulum is conveniently displayed graphically in the plane of its phase variable, angle and angular velocity. The characteristic points or curves in these diagrams are called “attractors" because irrespective of initial conditions, all trajectories are asymptotically attracted to them.

(iii) *Fast Fourier Transform* - Chaotic motion contains an infinite number of components and therefore Fourier analysis is also an important tool.

(iv) *Poincare-Map* - The Poincare section is obtained by removing one space dimension. In the case of three variables, the Poincare section is a plane of function and its derivative. The Poincare plane has to be chosen in such a way that trajectories will intersect it several times. If the motion is periodic (non chaotic), the trajectory will cross the Poincare plane repeatedly at the same point. Chaotic motion means that despite the fact that the motion is deterministic it never repeats itself, however there will be a dense set of points in the Poincare section filling a certain area of this plane. Poincare sections are useful when analyzing chaotic systems, as they make it easier to understand their dynamics. It is very important to construct Poincare sections intelligently to be able to visualize the dynamics of chaotic systems [27].

(v) *Lyapunov exponent* - The Lyapunov exponents are the average rate of contraction or expansion near the periodic orbit. Knowing how the local Lyapunov exponent varies in space allows one to identify regions of an attractor with good or poor predictability for small initial errors.

The Lyapunov exponents can reveal if indeed there is an exponential relationship between the flows of two very close initial conditions. In general, for an n-dimensional dynamical system, there are n Lyapunov exponents. To check for sensitivity of initial conditions, only the largest Lyapunov exponent is of interest. The method for finding this largest Lyapunov exponent is very similar to finding the Lyapunov exponent of a one-dimensional map which is explained next.

Assume P is a Poincar´e map of a 1-D system. Let z_0 and $z_0+\Delta z_0$ be two nearby initial points on the flow, not necessarily in steady state. After one iteration of a map the points are separated by

Δz_1 = P($z_0+\Delta z_0$) - P(z_0) $\approx \Delta z_0 P' z_0$

where $P' = \frac{dp}{dz}$. The local Lyapunov exponent λ at isz_0

$$\lambda = \ln|\frac{\Delta z_1}{\Delta z_0}| \approx \ln|P'(z_0)|$$

To obtain the global Lyapunov exponent, an average of the local Lyapunov exponent over much iteration must be taken

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \ln|\frac{\Delta Z_1}{\Delta Z_0}|$$

This is similar to calculating the eigenvalues of the linearized Poincar´e map Whereas eigenvalues are usually calculated at a point in state space, such as a fixed point, Lyapunov exponents are usually geometrically averaged along the orbit.

1.4 LAGRANGE EQUATION OF MOTION

Consider a mechanical system with 'n' degrees of freedom, locally represented by n generalized configuration (position) coordinates $q = (q_{1,} q_{2,} .... q_n)$.In classical mechanics the following equations of motion are derived

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \qquad i \in n \tag{1.8}$$

Here $T(q, \dot{q})$, with $\dot{q} = (\dot{q}_1, \dot{q}_2, \dot{q}_3, ...\dot{q}_n)$ the generalized velocities, denotes the total kinetic energy of the system, *Qi* are the generalized active forces, which act on the system. Usually the forces *Qi* are decomposed in to a part which are called conservative forces, i.e. forces that are derivable from potential energy, and a remaining part $Q^{e}{}_{i}$, $i \in n$, consisting of dissipative and generalized external forces.

$$Q_i = -\frac{\partial V}{\partial q_i}(q) + Q^{e}{}_{i} \qquad i \in n, \tag{1.9}$$

With *V(q)* being the potential energy function. Defining the Lagrangian function $L_0(q,\dot{q})$ as $T(q,\dot{q}) - V(q)$, arrives at the Euler Lagrange equations

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_i} - \frac{\partial L_0}{\partial q_i} = Q^{e}{}_{i}, \qquad i \in n, \tag{1.10}$$

From Eqn. (1.10) a control system is obtained by disregarding dissipative forces and interpreting the external forces $Q^{e}{}_{i}$ in Eqn. (1.9) as input or control variables u_i .

If some degree of freedom can be directly controlled, then control system obtains

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_i} - \frac{\partial L_0}{\partial q_i} = \begin{Bmatrix} u_i, i = 1, \ldots\ldots.k \\ 0, i = k+1 \ldots.n \end{Bmatrix} \tag{1.11}$$

With $u_1, \ldots\ldots, u_k$ are the controls.

In general, for a mechanical system of n degrees of freedom with a Lagrangian $L = L(q, \dot{q}, \mathrm{u})$ depending directly on *u*, in the absence of other forces, the equation of motion

$$\frac{d}{dt}\frac{\partial L(q,\dot{q},\mathrm{u})}{\partial \dot{q}_i} - \frac{\partial L(q,\dot{q},\mathrm{u})}{\partial q_i} = 0, \qquad i \in n, \tag{1.12}$$

Eqn. (1.11) can be regarded as special case of Eqn. (1.12) by taking in Eqn. (1.12) the Lagrangian

$$L(q,\dot{q},\mathrm{u}) = L_0(q,\dot{q}) + \sum_{m=1}^{n} q_m u_m \tag{1.13}$$

Eqn. (1.13) known as a Lagrangian control system [3].

1.5 HAMILTONIAN MECHANICS

Systems or models with no dissipation are called conservative systems or Hamiltonian systems. The term conservative means physical properties such as the angular momentum, total mechanical energy remain constant in time. Hamiltonian systems comprise a class of dynamical systems in which some quantity (typically energy) is constant along the system's trajectories [1, 2]. In the Hamiltonian formulation of classical mechanics, the time evolution of a system is described in terms of set of dynamical variable, which give position and momenta of the system [5]. There is another description of the dynamics of the system by means of generalized coordinates q_i and another quantity the so called generalized momenta p_i.

From the Lagrangian control system Eqn. (1.12), the generalized momenta can be defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q,\dot{q},u), \qquad i \in n, \tag{1.14}$$

The Hamiltonian function $H(q, p, u)$ as the Legendre transform of $L = L(q,\dot{q},\mathrm{u})$,

$$H(q, p, u) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q,\dot{q},u), \tag{1.15}$$

Where $\dot{q}$ and p are related by Eqn. (1.14).

It is clear that with Eqn. (1.14) and (1.15) the Euler-Lagrange Eqn. (1.12) transform in to the Hamiltonian equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial \dot{p}_i} \qquad , \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i \epsilon n \tag{1.16}$$

The quantity H is called the Hamiltonian. The Eqn.(1.16) known as Hamiltonian control systems. The advantage of Eqn. (1.16) in comparison with Eqn. (1.12) is that Eqn. (1.16) immediately constitutes a control system in standard state space variable [3].

1.6 GENERAL PROCEDURE TO GET THE HAMILTONIAN EQUATIONS

The procedure for receiving the Hamilton equations, can be shortly summarized in the following algorithm

1. Write the Lagrange function in Cartesian coordinates $L = T - V$
2. Choose the generalized coordinates $q = (q_{1,} q_{2,} ....q_n)$ according to the constraints.
3. Express the kinetic energy T and potential V energy by the generalized coordinates q and $\dot{q}$
4. Find the generalized momenta $p = (p_1, p_2, ...p_n)$ using the basic formulae $\dot{p}_i = \dfrac{\partial L}{\partial \theta_i}$.

5. Express $\dot{q}$ by p, q and write Hamiltonian as a function of p and q using the formulae $H = \sum p_i \dot{q}_i - L$. If our coordinates do not depend explicitly on time, then the Hamiltonian is just the integral of energy $H = T + V$

6. Write the canonical Hamiltonian equations.

1.7 OBJECTIVES

The main objective of the work is to see the dynamical behaviour of the multiple link pendula. The modeling of double and triple pendulum is obtained using Lagrangian and Hamiltonian based approach for lumped and distributed system. The simple pendulum shows periodic motion under free vibration whereas double to triple pendulum shows quasi periodic to chaotic behaviour. As the system shows the chaotic behaviour so all the tools for the measurement of chaos like Poincare', Lyapunov Exponent, Time series analysis, Fast Fourier Transform etc. have been discussed in detail. The mass and length dependent behavior on double pendulum is calculated. Also the effect of damping is also taken in to account. An optimal based LQR control approach is used for stabilizing the triple link inverted pendulum on cart.

1.8 OUTLINE OF THE CHAPTERS

Chapter 1 presents the introduction to system theory and basic equation for the modelling of the system. The Lagrange and Hamiltonian based approach is discussed for the dynamics of pendulum.

Chapter 2 presents the literature survey and what is final outcome of the literature survey.

Chapter 3 is focusing on the how to obtain the dynamics of SP, DP for lumped and distributed system.

Chapter 4 presents the dynamics and control for TP for lumped and distributed system.

Chapter 5 presents the results and discussion. Results on the mass and length dependent behaviour will be discussed in detail for double pendulum. The effect on the fundamental frequency has been shown when go from lumped to distributed system in case of Double and triple pendulum. Also the Stability output of TP has been shown.

Chapter 6 presents the conclusion and future scope

CHAPTER 2

LITERATURE SURVEY

The chapter describes the literature survey that being carried out in understanding the concept of dynamics and control of multiple pendula. Modelling is obtained with the help of Lagrangian and Hamiltonian Mechanics reported in the literature.

2.1 LITERATURE REVIEW

Modern development in the field of mechanics, mathematics and related numerical calculation technique allow more exact modelling of the real time dynamic phenomenon that are exhibited by various physical objects. Nonlinear dynamics is the study of time-evolving systems governed by equations where superposition fails [3]. The behavior of a non-linear system differs considerably from a linear system and non-linear systems are found to show periodic, a-periodic, quasi-periodic & chaotic dynamical behavior. These aspect of non-linearity are easily identified in dynamics of simple and multiple pendula systems. A mechanical pendulum with single or double Degree of Freedom (DoF) is widely used for validating control algorithms as well as dynamic model for variety of physical systems .The inverted pendulum IP is most widely used system from the control point of view and it has various application in the area of engineering and science [4]. The instability of the pendulum in its inverted state can be removed by applying rapid vertical oscillatory forcing to its pivot point.

This phenomenon had been discovered in early 20th century by Stephenson [5].Many research papers have discussed the oscillations of a damped driven Pendulum. Here are some examples; Blackburn et al [6] studied the stability and Hopf bifurcation in an inverted pendulum, Smith et al [7] investigated the behaviour of an inverted pendulum through experimental measurements, Kalmus [8] worked on a driven inverted pendulum experiment using a speaker, Michaelis [9] used an electric jigsaw to drive an inverted pendulum and study its behaviour through stroboscopic photos; Acheson et al [10] compared the stability of an inverted pendulum from theoretical models and experimental approaches. Vela work [11] shows that nonlinear time varying control systems can also be given an exponential representation means that intuition and analysis from linear control theory may provide the control engineer with the needed background to construct and analyze stabilizing controllers for nonlinear systems.

Fig. 2.1 is going to use for showing the study that being carried out the whole literature. The pendulum as a mechanical system can be applied as a free oscillation or forced oscillation. The modelling for double and triple pendulum under free oscillation has been obtained for lumped and distributed system. In certain literature [32-33] modelling is there but for forced oscillation. More mathematically, it can be thought of as the type of motion executed by a dynamical system containing a finite number of incommensurable frequencies [21]. No literature has been found for the pendulum under free oscillation in case of triple pendulum. Main challenges are in obtaining the exact modelling of the triple pendulum and for that Euler - Lagrange and Momenta based approach is used for lumped and distributed system.

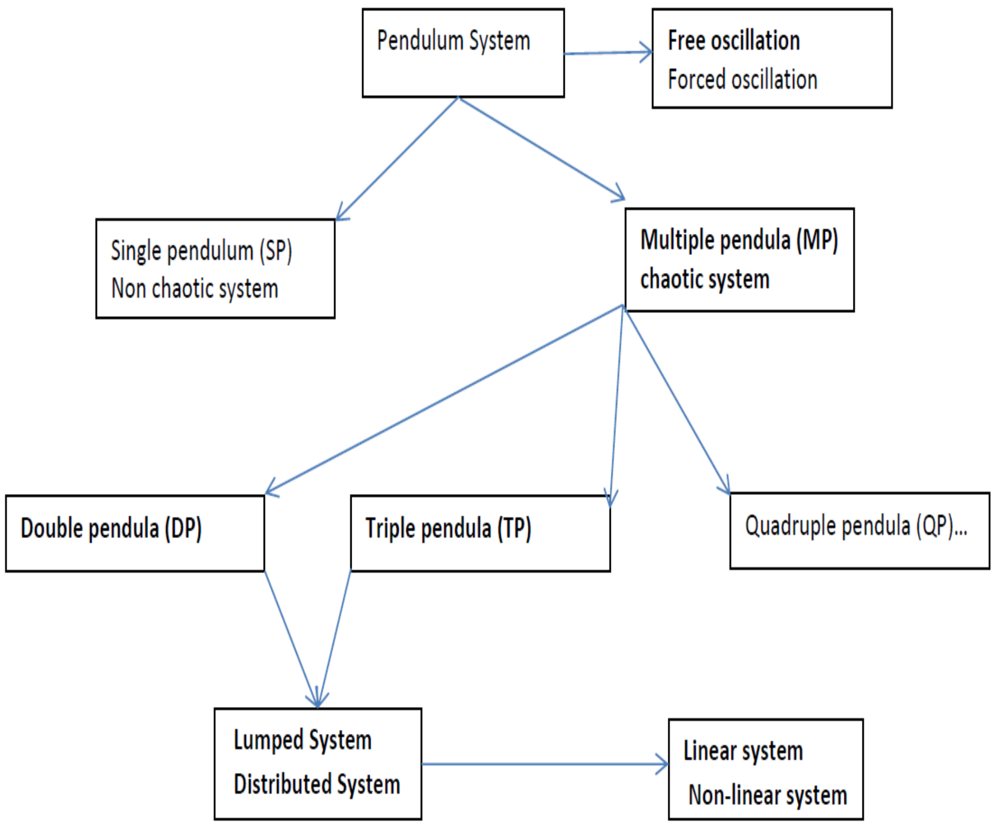


Fig. 2.1 Summery of the literature using Block Diagram

The dynamics of a double pendulum can be described with 4 variables, the two angles and their corresponding (angular) velocities, which span the four dimensional phase space of the system [13]. Kolmogorov-Arnold-Moser (KAM) tells us that at lower energies, the function is integrable (it has as many conserved quantities as there are degrees of freedom in the system) [14]. From the theoretical evidence, it has been hypothesize that the behavior of a double pendulum varies from regular motion at low energies, to chaos at intermediate energies, and back to regular motion at high energies [15].The notion of fixed points or equilibrium points in state space plays a key role in understanding the dynamics of nonlinear systems.

Control of mechanical systems is currently among one of the most active fields of research because of the diverse applications of mechanical systems in real-life. Though, the study of mechanical systems goes back to Euler and Lagrange in the 1700's, it was not until 1850's that mechanical control systems came to the picture in regulation of steam engines. On the other hand, theoretically challenging nature of analysis of the behavior of non-linear dynamical systems attracted many mathematicians to study the detail analysis of dynamic and control systems as a result, the efforts of engineers and scientists together led to creation of nonlinear dynamics theories [17, 18].

The inverted pendulum is a type of under actuated mechanical systems that have fewer control inputs than configuration variables. Underactuated systems [41] appear in a broad range of applications including, Aerospace Systems, Robotics, Marine Systems, Mobile Systems Flexible Systems and Locomotive Systems. The “under actuation" property of under-actuated systems is due to the following four reasons: (i) dynamics of the system(e.g. aircraft, spacecraft, helicopters, underwater vehicles, locomotive systems with-out wheels), ii) by design for reduction of the cost or some practical purposes (e.g. satellites with two thrusters and flexible -link robots), iii) actuator failure (e.g. in a surface vessel or aircraft), iv) imposed artificially to create complex low-order nonlinear systems for the purpose of gaining insight in control of high-order Underactuated systems[16].

A Double Pendulum (DP) is an interesting dynamical system since it shows chaotic motion [19, 20, 25]. Notably, experimental and numerical studies have been carried out of such a system considering the change in initial value of amplitude and angular velocity of the DP [20]. It is observed that a DP shows periodic behaviour at low energy, transforms to quasi-periodicity at intermediate energy level and chaos at higher energy level finally again periodic motion as energy of the system increases further [21]. Despite significant study on the chaotic dynamics of the DP, there is no reported study in literature how the mass and length of a DP influences its chaotic behaviour. Such a study is important from controlling and optimizing dynamical systems based on double pendulum for instance double arm robots [20, 22,23].In the fast Fourier transform (FFT) analysis of the nonlinear system quasi-periodic scenario comes in to the picture because two or more frequencies involves in the dynamics of the system. The periodic doubling behaviour leads to bifurcation and chaos.

The state space dimensionally is determined by the number of variables needed to specify the dynamical state of the system. A state space and a rule for following the evolution of trajectories starting at various initial conditions constitute what is called a dynamical system [26]. The mathematical theory of such systems is called dynamical system theory.

Chaos is a physical phenomenon in which a dynamical system shows random motion and the results depends on initial conditions. Chaos is basically a manifestation of non-linearity in the dynamical system. However, all non-linear systems do not show chaotic behaviour. A condition for chaos is that the differential equations governing the dynamical system should contain non-linear terms. Moreover, it should be expressible in at least three or more phase variables [20]. Lyapunov exponent is generally used to characterize the chaos of a dynamical system. For instance, if the average value of Lyapunov exponent is negative then the system is non chaotic. Lyapunov exponent if positive, then it shows chaotic behaviour. Linear stability analysis of the DP shows that its two natural frequencies depend on mass and length of each pendulum in the system [19]. Largest Lyapunov Exponents gives us the information on the divergence of two close trajectories [23]. MATLAB software is used to evaluate the Lyapunov exponent. Further it is to be noted that natural frequency of a pendulum also depends on the initial condition of oscillation for instance amplitude. Numerical integration must be used to complement analytical work because non-linear differential equations can almost never be exactly solved analytically and can be solved using the Runge -Kutta method [24].

The dynamics for Double Pendulum as a lumped system [26] is obtained then go for the distributed DP. Lumped systems are those system in which mass is concentrated at one point where as in case of distributed system mass is distributed uniformly. Also the mass and Length dependent behaviour also studied in detail [28]. As going further to Triple Pendulum (TP) then dynamic analysis is further challenging task .For the TP for lumped and distributed system study has been done and experimentally verified. In 2001, Laser Interferometer Gravitational Wave Observatory (LIGO) was slated to update their facility by housing each mirror on the lowest bob of a quadruple pendulum to reduce thermal noise. These developments have produced scientific interest in the dynamics of triple pendulum [30, 31]. The study of a triple link system is a highly non-linear, multi-variable, higher order, unstable system can contribute to the development of walking robots, flexible space structures, and automatic aircraft landing system, biped locomotive machines since it can be considered a simplified model of the human standing on one leg.

In the paper, a triple link inverted pendulum mounted on a cart that can move horizontally is controllable and can be stabilized in the upward position with a single control input [32, 33]. A design of robust control system to balance a TP mounted on a cart was considered in Medrano Cerda in 1997. The controller design is there based on discrete time linear regulator theory that was implemented by robust observer. The relative stability and disturbance were investigated using Frequency response method [34]. A single input feedback controller for a TP was designed in 1998 by Elthomy using nonlinear optimization technique turned out to be very effective. Based on the gains generated by such nonlinear optimization, a custom built TP was successfully stabilized in experimental rig [33]. In 2003 the controllability of damped TP was investigated. The work focuses on cancelling pole and zero that appears in the transfer function of an uncontrollable system [36].

A lot of work has been done on stabilizing the TP [42, 43]. Optimal based control is one of the simplest technique for stabilizing all the parameter of TP [38]. A Linear Quadratic Regulator (LQR) based control technique is used for controlling the Triple link inverted pendulum system [37]. The performance index or cost function is minimized in LQR control technique. Necessary condition for optimality like Hamiltonian, optimal control, state and costate system, closed loop optimal control, Matrix Differential Ricatti equation keep in for obtaining the control law [37, 42].

The dynamic modelling and behaviour of TP for lumped and distributed system has been developed. Experimentally for lumped triple pendulum system some application has been discussed [45, 48] but not experimentally verified for the distributed one. The chaos on DP and TP system has many applications [49, 50] and details are given in robot manipulator.

2.2 **CONCLUSION FROM LITERATURE SYRVEY**

As seen from the literature survey that the various article has been presented on the concept of dynamics and control simple and double pendulum under free and forced oscillation. As far as triple pendulum distributed system is concern there is not much literature available. In some of the paper for TP dynamics is given for lumped system for not for distributed system under free oscillation. It is founded in the literature that detailed nonlinear dynamic analysis of TP is not available for lumped and distributed system with experimental verification. The effect of damping is also not found in the literature. In the literature also it is not discussed that what changes occur in the dynamics and Frequency when moving from lumped to distributed system. It is also clear that effect of mass and length on the pendulum not found. As the multiple link pendulum shows chaotic behaviour so main emphasis on the parameter for measuring chaos like Poincare Map, Lyapunov Exponent, Time series analysis. For controlling the Triple Link Inverted Pendulum LQR based technique is used for controlling all the parameter of TLIP.

CHAPTER 3

DYNAMICS OF SIMPLE AND DOUBLE PENDULUM

The chapter describes the mathematical modelling for obtaining the dynamics of simple and double pendulum. Lagrangian and Hamiltonian based approach is used for obtaining the governing differential equations of simple and double pendulum.

3.1 THE SIMPLE PENDULUM

The simple pendulum consists of a point mass *m*, attached to an infinitely light rod of length l .There is no damping force. The angle θ is the angular displacement of the pendulum from the vertical position.

Fig. 3.1(a) Fig. 3.1(b)

Fig. 3.1 The simple pendulum

A simple pendulum is a two degree of freedom system as motion takes place in the (x, y) plane only but it can reduce to easily one degree of freedom system. Forall the time the rod length can be represented by $\sqrt{x^2 + y^2} = l$. As

$y=\sqrt{l^2-x^2}$, so y dependent on x coordinate only as length l is a constant term.

The coordinate position of the pendulum

$$x=l\sin\theta,\; y=l\cos\theta \tag{3.1}$$

The time derivative of the Eqn. (3.1)

$$\dot{x}=l\dot{\theta}\cos\theta,\; \dot{y}=-l\dot{\theta}\sin\theta \tag{3.2}$$

The Lagrange (L) is defined as the difference between Kinetic energy T and Potential Energy V

$$L=T-V=\frac{1}{2}ml^2\dot{\theta}^2-mgl(1-\cos\theta) \tag{3.3}$$

The generalized momentum can be obtained from Eqn. (3.3)

$$p=\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta} \tag{3.4}$$

From Eqn. (3.4) $\dot{\theta}$ can be expressed by p

$$\dot{\theta}=\frac{p}{ml^2} \tag{3.5}$$

Hamiltonian as a function of p and θ

$$H=p\dot{\theta}-L \tag{3.6}$$

Substituting the Eqn. (3.3) and Eqn.(3.4) in (Eqn. 3.6) Hamiltonian (H)

$$H=\frac{p^2}{2ml^2}+mgl(1-\cos\theta) \tag{3.7}$$

Angular Velocity and Momentum can be calculated as follows

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{p}{ml^2} \;,\; \dot{p} = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \tag{3.8}$$

Differentiating the first equation in system Eqn. (3.8) and substituting in the second, the Eqn. (3.9) can be obtained as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{3.9}$$

Because of the term $\sin\theta$ the above formula is nonlinear in nature. Therefore solving the Eqn. (3.9) explicitly is not easy. However for small value of θ, $\sin\theta$ reduces to θ then Eqn. (3.9) changes to $\ddot{\theta} + \frac{g}{l}\theta = 0$, where $\omega_0^2 = g/l$, is ω^0 called fundamental or natural frequency of the system.

3.2 PHASE PLANE

A convenient way to understand the qualitative dynamics of dynamical system with state x in $\mathbb{R}^2$ is the Phase Portrait of the system. Phase plane is a technique of studying the behaviour of nonlinear system. Each value of θ gives a closed orbit of constant energy. A Phase Portrait is a graphical representation of the trajectories of a dynamical system in the phase plane.

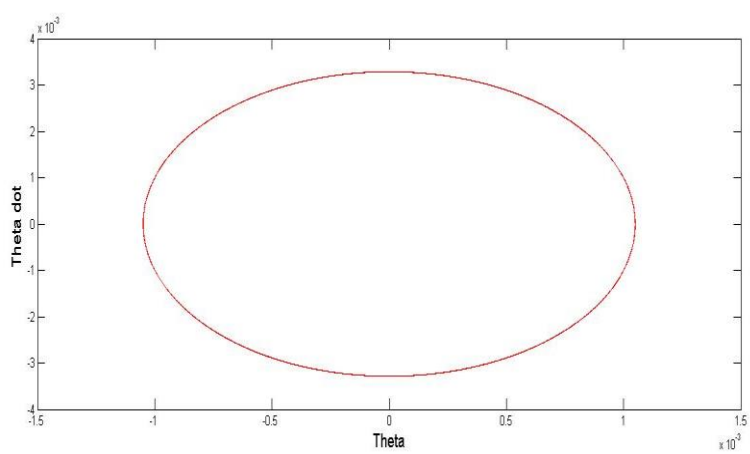


Fig. 3.2 Phase portrait of simple pendulum

3.3 VECTOR FIELD OF SIMPLE PENDULUM

Vector field gives a vector pointing in the direction of the velocity at every point in phase space. Vector field having the following Properties.

(1) The vector at any point in the state space is unique.

(2) Two vector field lines can never intersect – except at the points where the magnitude of the vector $\dot{x}$ becomes zero, that is, at the equilibrium points.

As one moves away from the equilibrium point, the local linear approximation no longer remains valid. As a result, the lines that started as eigenvectors in the neighborhood of an equilibrium point will no longer remain straight lines. These curved lines are called invariant manifolds, which have the property that if an initial condition is placed on the manifold, its future evolution also remains on the same manifold. The stable and unstable eigenvectors at an equilibrium point are locally tangent to these manifolds. If the state point approaches an equilibrium point along an invariant manifold, it is called a stable manifold, and if the state moves away from an equilibrium point along an invariant manifold, it is called an unstable manifold [29]. Vector field of simple pendulum is oscillatory in nature and is towards the equilibrium point.

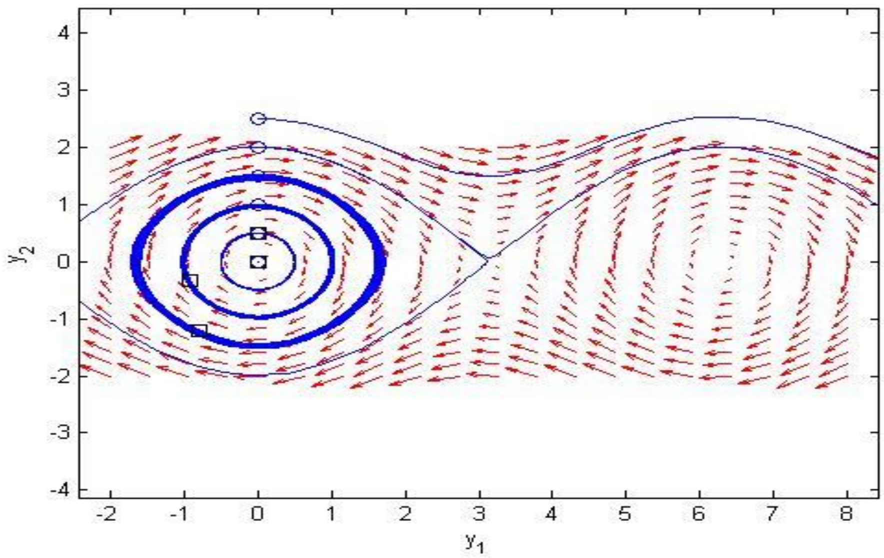


Fig. 3.3 Vector Field of simple Pendulum

3.3.1 THE FORCED DAMPED DRIVEN PENDULUM

The single pendulum equation is nonlinear in nature. Despite its nonlinearity it does not show any chaotic behaviour. So the question is what conditions should be the sufficient condition so the nonlinear equation show the creation of chaos. A dynamical system with n ≥ 3 autonomous first order differential equations i.e. phase space must be at least three dimensional shows chaotic behaviour.

3.3.2 EQUATION OF MOTION FOR FORCED DAMPED DRIVEN PENDULUM

The equation of motion becomes

$$\ddot{\theta} + b\dot{\theta} + \omega_n^2 \sin\theta = f_d \sin\omega_d t \tag{3.10}$$

Where f_d is driving force, ω_n is natural frequency, b is friction strength and ω_d is driving frequency. Eqn. (3.10) is the second order non autonomous differential equation. In order to analyze its properties; it can reduce to autonomous first order equations with variables.

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \\ x_3 &= \omega_d t, \end{aligned} \tag{3.11}$$

In these variables equations (3.11) takes the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 - \omega_n^2 \sin x_1 + f_d \sin x_3 t \\ \dot{x}_3 &= \omega_d \end{aligned} \tag{3.12}$$

The Eqn. (3.12) is 3-Dimensional system of autonomous ordinary differential equations. The condition of the appearance of chaos that dimension of phase space n $\geq$ 3 is satisfied. Hence it follows that Eqn. (3.12) is a good candidate to observe the chaotic behaviour.

3.4 VECTOR FIELD OF FORCED DAMPED DRIVEN PENDULUM

A vector field in the plane can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. The vector field of damped driven pendulum is shown in fig. 3.4. It is clear from the fig. 3.4 that vector fields are irregular in nature which leads to chaotic behaviour.

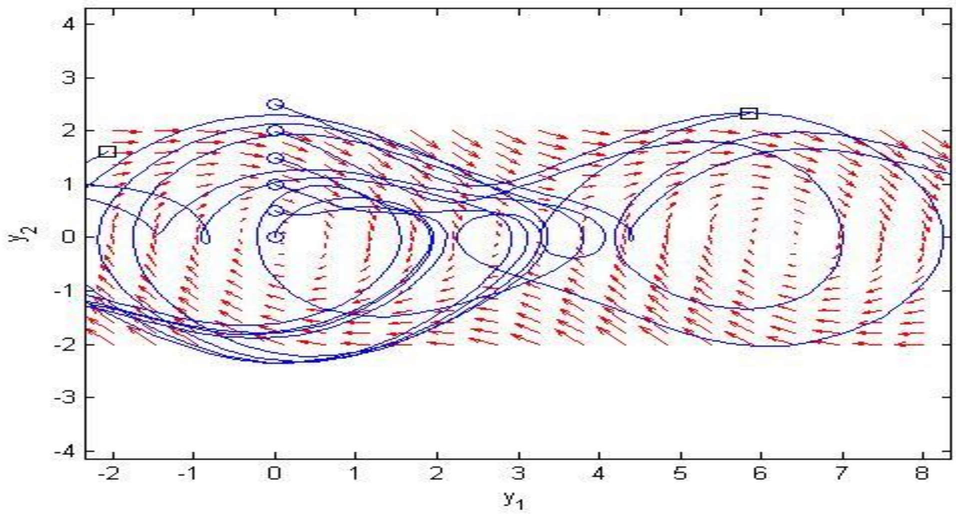


Fig. 3.4 Vector field of damped driven pendulum

3.5 THE POINCARE MAPPING

The Chaos is related to the sensitivity to initial conditions and the values of coordinates that never repeats itself. There are various techniques for the study of chaos but the method which simplifies the phase portrait is called Poincare mapping. Poincare considered the complex trajectories in phase space. Henri Poincare. Imagine a surface as shown in Fig. 3.5, called the Poincar'e section, at a suitable place in the state space such that the orbit intersects it.

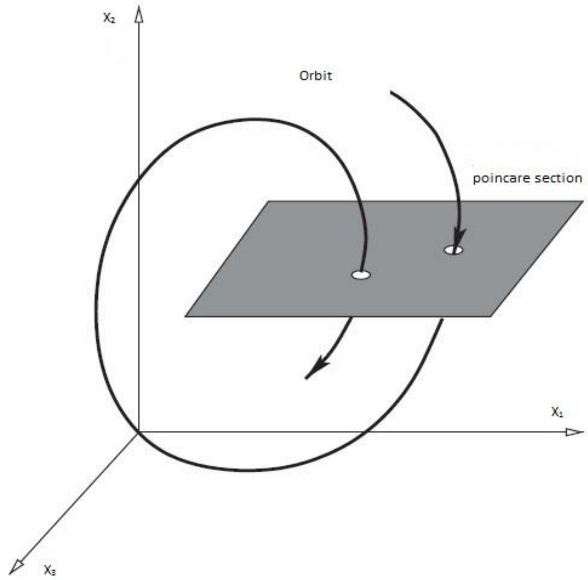


Fig. 3.5 The Poincare section intersection orbit

Let us consider an Autonomous system consisting of ODE of first order

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, x_3),\\
\dot{x}_2 &= f_2(x_1, x_2, x_3),\\
\dot{x}_3 &= f_3(x_1, x_2, x_3).
\end{aligned} \tag{3.13}
$$

The Poincar´e mapping reduces the trajectory of n dimensional phase space to n-1 dimensional mapping. In Eqn. (3.13) trajectory of 3 dimensional phase space reduces to 2 dimensional phase space. Each time when the trajectory pierces Poincar´e section in downward direction, this point is marked on the cross section plane. The First point and second point represent two intersections (pierces). First Point can be treated as an initial condition of the Eqn. (3.13) to determine second point.

3.6 POINCARE MAPPING OF FORCED DAMPED PENDULUM

Poincare mapping is applied for the forced damped pendulum described by Eqn. (3.10) in three-dimensional phase space (x_1, x_2, x_3).The third variable is proportional to time $x_3 = \omega_d t$.

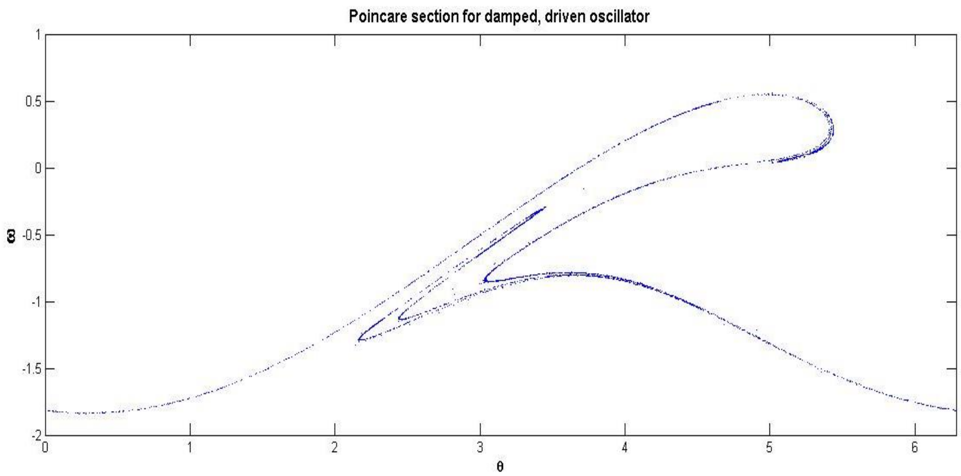


Fig. 3.6 Poincare section, ω_n = 1, b = 0.5, f_d = 1.2, ω_d = 2/3

The mathematical model and dynamics of simple and forced Pendulum is obtained. In case of simple pendulum under free oscillation motion is regular whereas in the case of forced oscillation motion shows some kind of irregularity that can be seen from the vector field analysis. When simple pendulum is inverted then it became inverted pendulum which is inherently unstable in nature. Inverted Pendulum (IP) is a very common and interesting nonlinear system in the control applications. Unstable IP system is usually used to test performance of the different control algorithms.

3.7 THE DOUBLE PENDULUM

In case of Double Pendulum (DP), two simple pendulums are attached to each other. The length and mass of the first link is $l_{1,}$ m_1 and for the second link is $l_{2,}$ $m_{2.}$ The angle from the first and second link is θ_1 and θ_2 respectively.

3.7.1 MODELLING OF DOUBLE PENDULUM

The DP has two degree of freedom system described by θ_1 and θ_2 .This DP shows periodic and quasi-periodic motions coexisting together with chaotic motions. A similar procedure is followed to derive the system of non-linear equations. Assuming the pivot point is at the origin of the coordination system with the horizontal (x-axis) and the vertical (axis). The DP is assumed to be located in the fourth quadrant of the coordinate system.

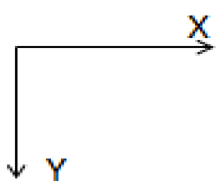


Fig. 3.7 (a)

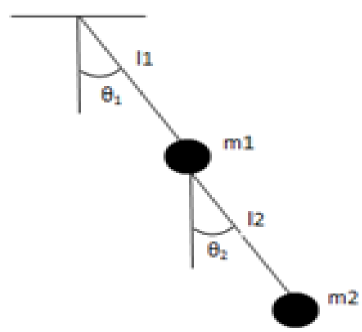


Fig 3.7 (b)

Fig3.7 Double Pendulum with Lumped Mass

3.7.2 EQUATION OF MOTION

The coordinate position of top pendulum is given by

$$x_1 = l_1 \sin(\theta_1)\,,\; y_1 = -l_1 \cos(\theta_1) \tag{3.14}$$

The coordinate position of bottom pendulum

$$x_1 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2),\;\; y_1 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \tag{3.15}$$

And the corresponding angular velocity is given by the equations

Time derivative of the Eqn. (3.14)

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1\,,\; \dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta}_1 \tag{3.16}$$

Time derivative of the Eqn. (3.15)

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2 \tag{3.17}$$

$$\dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta}_1 + l_2 \sin(\theta_2)\dot{\theta}_2 \tag{3.18}$$

Kinetic energy of the double pendulum in Fig.3.7 is given by the following equation

$$K = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}(m_2)l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{3.19}$$

And the potential energy is

$$V = -(m_1 + m_2)gl_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \tag{3.20}$$

The Lagrangian $L = K - V$ of two simply connected pendulums is given by the following equation

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 +$$
$$m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos\theta_1 \tag{3.21}$$
$$+ m_2 g l_2 \cos\theta_2$$

The Lagrange equations of motion of the system can be obtained from the Eqn.

(3.22) and Eqn. (3.21)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0,\ i = 1,2 \tag{3.22}$$

First Lagrange equation of motion

$$\begin{aligned}&(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)\\ &+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0\end{aligned} \tag{3.23}$$

Second Lagrange Equation of Motion

$$\begin{aligned}&m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\\ &+ m_2 g l_2 \sin\theta_2 = 0\end{aligned} \tag{3.24}$$

Eqn. (3.23) and Eqn. (3.24) can be written in the following form

$$b_{11}\ddot{\theta}_1 + b_{12}\ddot{\theta}_2 + b_1 = 0 \tag{3.25}$$

$$b_{21}\ddot{\theta}_1 + b_{22}\ddot{\theta}_2 + b_2 = 0 \tag{3.26}$$

Where $b_{11} = (m_1 + m_2)l_1^2$,

$b_{12} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2)$,

$$b_1 = m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 ,$$

$$b_{22} = m_2 l_2^2 \,,$$

$$b_{21} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \,,$$

$$b_2 = -m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2$$

And the equation of momenta p_i of the DP is given by $p_i = \partial L / \partial \dot{\theta}_i$ where $i = 1, 2$

$$p_1 = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \qquad (3.27)$$

$$p_2 = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \qquad (3.28)$$

And the Hamiltonian $H = K + V$ of the DP is given by

$$\begin{aligned} H &= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &\quad -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned} \qquad (3.29)$$

Above equations can be used for the dynamic analysis of ouble pendulum.

3.8 MODELLING OF DOUBLEPENDULUM WITH DISTRIBUTED MASS

If $c1$ and $c2$ are the mid points of the two length l_1 and l_2 of distributed mass m_1 and m_2 .The value of $c1$ and $c2$ are respectively as follows

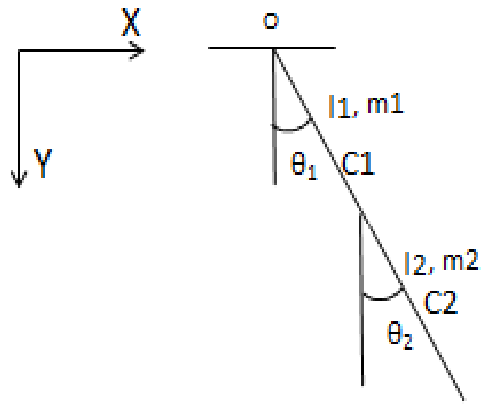


Fig.3.8 Double pendulum as a distributed system

$$c1 = (\frac{l_1}{2}\sin\theta_1, -\frac{l_1}{2}\cos\theta_1), \tag{3.30}$$

$$c2 = (l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2, -l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2) \tag{3.31}$$

The Kinetic Energy of the system

$$K.E. = \sum_{i=1}^{2}\frac{I_{ci}\dot{\theta}_i^2}{2} + \frac{m_i v_i^2}{2} \tag{3.32}$$

And the Potential energy

$$P.E. = -\frac{m_1 g l_1 \cos\theta_1}{2} - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2 / 2) + \frac{m_1 g l_1}{2} + m_2 g(l_1 + \frac{l_2}{2}) \tag{3.33}$$

The Lagrange of the system is defined as

$L = K.E. - P.E.$, substituting the value of K.E. and P.E. from Eqn. (3.32 and 3.33) respectively leads to Eqn. (3.34)

$$L=\frac{I_{c1}\dot{\theta}_1^2}{2}+\frac{m_1 l_1^2\dot{\theta}_1^2}{8}+\frac{I_{c2}\dot{\theta}_2^2}{2}+\frac{m_2}{2}\{l_1^2\dot{\theta}_1^2+\frac{l_2^2\dot{\theta}_2^2}{4}+l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\}$$
$$+\frac{m_1 g l_1\cos\theta_1}{2}+m_2 g(l_1\cos\theta_1+l_2\cos\theta_2/2)-\frac{m_1 g l_1}{2}-m_2 g(l_1+\frac{l_2}{2}) \tag{3.34}$$

Where $I_{ci}=\frac{m_i l_i^2}{12}$, is the moment of inertia of the system.

The Lagrange equation of motion can be obtained using the following formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_i}\right)-\frac{\partial L}{\partial\theta_i}+\frac{\partial Q}{\partial\theta_i}=0\,,\qquad i=1,2 \tag{3.35}$$

Here Q is Rayleigh damping.

$$Q=\frac{c_1}{2}\dot{\theta}_1^{\,2}+\frac{c_2}{2}(\dot{\theta}_1-\dot{\theta}_2)^2 \tag{3.36}$$

The matrix of the following form can be obtained from the eqn. (3.35)

$$c_{11}\ddot{\theta}_1+c_{12}\ddot{\theta}_2+d_1=0\ \&\ c_{21}\ddot{\theta}_1+c_{22}\ddot{\theta}_2+d_2=0 \tag{3.37}$$

Where coefficients are defined as follows

$c_{11}=I_{c1}+m_1 l_1^2/4+m_2 l_1^2,$

$c_{12}=m_2 l_1 l_2\cos(\theta_1-\theta_2)/2,$

$c_{21}=m_2 l_1 l_2\cos(\theta_1-\theta_2)/2,$

$c_{22}=I_{c2}+m_2 l_2^2/4.$

$$d_1 = -m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)/2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)/2 +$$
$$m_1 g l_1 \sin\theta_1/2 + m_2 g l_1 \sin\theta_1 + (c_1 + c_2)\dot{\theta}_1 - c_2 \dot{\theta}_2$$

$$d_2 = -m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)/2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)/2$$
$$-m_2 g l_2 \sin\theta_2/2 - c_2(\dot{\theta}_1 - \dot{\theta}_2)$$

3.9 STABILITY ANALYSIS OF DP FOR LUMPED SYSTEM

By separating the $\ddot{\theta}$ and θ term Eqn. (3.25 and 3.26) can be written as follows

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\} \tag{3.38}$$

Where

$$M = \begin{bmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \end{bmatrix}, \quad K = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For linear stability analysis assuming $\sin\theta = \theta, \cos\theta = 1$

$$M = \begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_2^2 & m_2 l_1 l_2 \end{bmatrix}, K = \begin{bmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}, \{\theta\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

The Eigenvalue of the Eqn. (3.38) is

$|-\lambda[M] + [K]| = 0$, Assuming $m_1 = m_2 = m, l_1 = l_2 = l$

$$\lambda^2 - 4g\lambda + 2g^2 = 0$$

$$\lambda_1 = 36.57, \lambda_2 = 2.62 \tag{3.39}$$

3.10 STABILITY ANALYSIS OF DP FOR DISTRIBUTED SYSTEM

By separating the $\ddot{\theta}$ and θ term Eqn. (3.37) can be written as follows

$$[M]\{\ddot{\theta}\}+[K]\{\theta\}=\{0\} \tag{3.40}$$

Where

$$M=\begin{bmatrix} c_{11},c_{12} \\ c_{21},c_{22} \end{bmatrix}, \quad K=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

For linear stability analysis assuming $\sin\theta=\theta, \cos\theta=1$

$$M=\begin{bmatrix} I_{c1}+\dfrac{m_1 l_1^2}{4}+m_2 l_1^2 & \dfrac{m_2}{2} l_1 l_2 \\ \dfrac{m_2}{2} l_1 l_2 & I_{c2}+\dfrac{m_2 l_2^2}{4} \end{bmatrix}, \{\theta\}=\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$K=\begin{bmatrix} \dfrac{m_1 g l_1}{2}+m_2 g l_1 & 0 \\ 0 & \dfrac{m_2 g l_2}{2} \end{bmatrix}$$

The eigenvalues of the equation (3.40) is

$|-\lambda[M]+[K]|=0$ Assuming $m_1=m_2=m, l_1=l_2=l$

$$|-\lambda[M]+[K]|=m\left|\begin{pmatrix} \dfrac{3}{2}g-\dfrac{4}{3}\lambda & \dfrac{-\lambda}{2} \\ -\dfrac{\lambda}{2} & \dfrac{g}{2}-\dfrac{\lambda}{3} \end{pmatrix}\right|$$

That is

$$7\lambda^2 - 42g\lambda + 27g^2 = 0$$

$$\lambda_1 = 51.62, \lambda_2 = 7.17$$

Both the Eigen values are positive hence the system will be unstable in nature.

3.11 CHAPTER SUMMARY

The chapter describes the modelling of single and double pendulum using energy based approach. The Modelling of double Pendulum is obtained for lumped and distributed system and final modelling is written in matrix form. The stability analysis also has been done.

CHAPTER 4

DYNAMICS OF TRIPLE PENDULUM

The chapter describes the modeling of triple pendulum for lumped and distributed system. The Euler-Lagrange and Lagrange formulation were used for the derivation of governing differential equations from two approaches.

4.1 MODELLING OF TRIPLE PENDULUM

The same method will be followed for formulating the Triple Pendulum (TP), as followed for Double Pendulum (DP).Triple pendulum is a three degree of freedom system having many applications in real life for instance robot manipulators, human body etc.

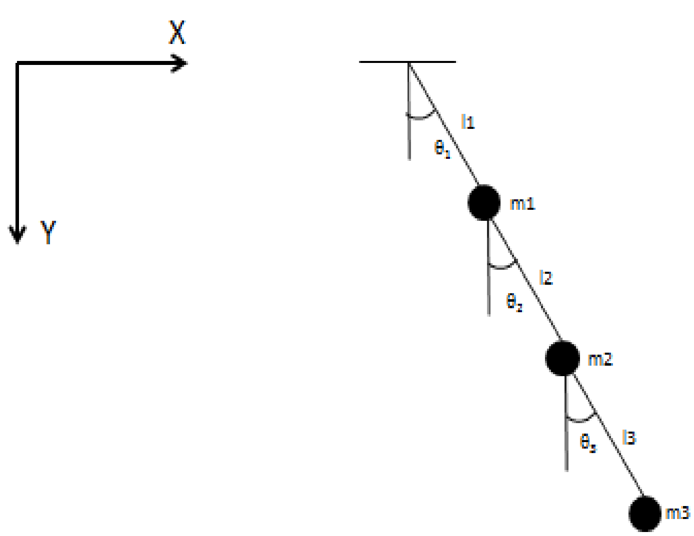


Fig. 4.1 Triple Pendulum system with lumped mass

The coordinate position of top link

$$x_1 = l_1 \sin(\theta_1),\ y_1 = l_1 \cos(\theta_1), \tag{4.1}$$

The coordinate position of middle link

$$x_1 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2),\ y_1 = l_1 \cos(\theta_1) + l_2 \cos(\theta_2) \tag{4.2}$$

The coordinate position of bottom link

$$x_1 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) + l_3 \sin(\theta_3),\ y_1 = l_1 \cos(\theta_1) + l_2 \cos(\theta_2) + l_3 \cos(\theta_3) \tag{4.3}$$

The time derivative of the Eqn. (4.1)

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1,\ \dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1 \tag{4.4}$$

The time derivative of the Eqn. (4.2)

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2 \tag{4.5}$$

$$\dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1 - l_2 \sin(\theta_2)\dot{\theta}_2 \tag{4.6}$$

The time derivative of the Eqn. (4.3)

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2 + l_3 \cos(\theta_3)\dot{\theta}_3 \tag{4.7}$$

$$\dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1 - l_2 \sin(\theta_2)\dot{\theta}_2 - l_3 \sin(\theta_3)\dot{\theta}_3 \tag{4.8}$$

The kinetic energy of the triple pendulum of Fig. 4.1 is given by the following equation

$$T(Q)=\frac{1}{2}(m_1+m_2+m_3)l_1^2\dot{\theta}_1^2+\frac{1}{2}(m_2+m_3)l_2^2\dot{\theta}_2^2+\frac{1}{2}m_3l_3^2\dot{\theta}_3^2+$$
$$m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+m_3(l_1l_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3)+l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{4.9}$$
$$+l_2l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3))$$

And the potential energy is

$$V(Q)=(m_1+m_2+m_3)gl_1(\cos\theta_1-1)+(m_2+m_3)gl_2(\cos\theta_2-1)+$$
$$m_3gl_3(\cos\theta_3-1) \tag{4.10}$$

Lagrangian (L) for three serially connected inverted pendulums is given by the following Eqn. (4.11)

$$L=T-V=$$
$$\frac{1}{2}(m_1+m_2+m_3)l_1^2\dot{\theta}_1^2+\frac{1}{2}(m_2+m_3)l_2^2\dot{\theta}_2^2+\frac{1}{2}m_3l_3^2\dot{\theta}_3^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{4.11}$$
$$+m_3(l_1l_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3)+l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+l_2l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3))$$
$$-(m_1+m_2+m_3)gl_1(\cos\theta_1-1)-(m_2+m_3)gl_2(\cos\theta_2-1)-(m_3)gl_3(\cos\theta_3-1)$$

First equation of momenta corresponding to $\dot{\theta}_1$

$$p_1=\frac{\partial L}{\partial\dot{\theta}_1}=(m_1+m_2+m_3)l_1^2\dot{\theta}_1++m_2l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2)+$$
$$m_3(l_1l_3\dot{\theta}_3\cos(\theta_1-\theta_3)+l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2)) \tag{4.12}$$

Second equation of momenta corresponding to $\dot{\theta}_2$

$$p_2=\frac{\partial L}{\partial\dot{\theta}_2}=(m_2+m_3)l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$
$$+m_3(l_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2)+l_2l_3\dot{\theta}_3\cos(\theta_2-\theta_3)) \tag{4.13}$$

Third equation of momenta corresponding to $\dot{\theta}_3$

$$p_3 = \frac{\partial L}{\partial \dot{\theta}_3} = m_3 l_3^2 \dot{\theta}_3 + m_3 l_1 l_3 \dot{\theta}_1 \cos(\theta_1 - \theta_3) + m_3 l_2 l_3 \dot{\theta}_2 \cos(\theta_2 - \theta_3) \quad (4.14)$$

First equation of angular momenta

$$\dot{p}_1 = \frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \sin(\theta_1 - \theta_3) - m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2 + m_3) g l_1 \sin\theta_1 \quad (4.15)$$

Second equation of angular momenta

$$\dot{p}_2 = \frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) + m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_2 + m_3) g l_2 \sin\theta_2 \quad (4.16)$$

Third Equation of angular momenta

$$\dot{p}_3 = \frac{\partial L}{\partial \theta_2} = m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \sin(\theta_1 - \theta_3) + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) + m_3 g l_3 \sin\theta_3 \quad (4.17)$$

Hamiltonian is the total Energy of the system is given by combining Eqn. (4.9) and Eqn. (4.10)

$$\begin{aligned} H = {} & \frac{1}{2}(m_1 + m_2 + m_3) l_1^2 \dot{\theta}_1^2 + \frac{1}{2}(m_2 + m_3) l_2^2 \dot{\theta}_2^2 + \frac{1}{2} m_3 l_3^2 \dot{\theta}_3^2 + \\ & m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_3 (l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \cos(\theta_1 - \theta_3) + \\ & l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3)) + (m_1 + m_2 + m_3) g l_1 \cos\theta_1 \\ & + (m_2 + m_3) g l_2 \cos\theta_2 + m_3 g l_3 \cos\theta_3 \end{aligned} \quad (4.18)$$

4.2 MODELLING OF TRIPLE LUMPED PENDULUM USING EULER LAGRANGE APPROACH

The same approach has been followed for formulating triple pendulum as was for double inverted pendulum. Lagrangian has been to derive the following equation.

Lagrangian for three serially connected inverted pendulums is given by the equation (4.11).Further modifying the above equations using the standard Euler-Lagrange equation for three degree of freedom as

$$
\begin{aligned}
& c_{11}\ddot{\theta}_1 + c_{12}\ddot{\theta}_2 + c_{13}\ddot{\theta}_3 + d_1 = 0 \\
& c_{11} = (m_1 + m_2 + m_3) l_1^2; \\
& c_{12} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2) + m_3 l_1 l_2 \cos(\theta_1 - \theta_2) \\
& c_{13} = m_3 l_1 l_3 \cos(\theta_1 - \theta_3) \\
& d_1 = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \sin(\theta_1 - \theta_3) + \\
& m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
& - m_3 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) - m_3 l_1 l_3 \dot{\theta}_3 (\dot{\theta}_1 - \dot{\theta}_3) \sin(\theta_1 - \theta_3) \\
& + (m_1 + m_2 + m_3) g l_1 \sin\theta_1
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
& c_{21}\ddot{\theta}_1 + c_{22}\ddot{\theta}_2 + c_{23}\ddot{\theta}_3 + d_2 = 0 \\
& c_{21} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2) + m_3 l_1 l_2 \cos(\theta_1 - \theta_2); \\
& c_{22} = (m_2 + m_3) l_2^2 \\
& c_{23} = m_3 l_2 l_3 \cos(\theta_2 - \theta_3) \\
& d_2 = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) \\
& - m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
& - m_3 l_2 l_3 \dot{\theta}_3 (\dot{\theta}_2 - \dot{\theta}_3) \sin(\theta_2 - \theta_3) - m_3 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
& + (m_2 + m_3) g l_2 \sin\theta_2
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
& c_{31}\ddot{\theta}_1 + c_{32}\ddot{\theta}_2 + c_{33}\ddot{\theta}_3 + d_3 = 0 \\
& c_{31} = m_3 l_1 l_3 \cos(\theta_1 - \theta_3) \\
& c_{32} = m_3 l_2 l_3 \cos(\theta_2 - \theta_3) \\
& c_{33} = m_3 l_3^2 \\
& d_3 = -m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \sin(\theta_1 - \theta_3) - m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) \\
& - m_3 l_1 l_3 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_3) \sin(\theta_1 - \theta_3) - m_3 l_2 l_3 \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_3) \sin(\theta_2 - \theta_3) \\
& + m_3 g l_3 \sin\theta_3
\end{aligned}
\tag{4.21}
$$

4.3 LINEAR STABILITY OF TRIPLE LUMPED PENDULUM

Combining above three equations (4.19 – 4.21) and linearizing about the vertical position of all three pendulum as

$$
\begin{aligned}
&c_{11}\ddot{\theta}_1 + c_{12}\ddot{\theta}_2 + c_{13}\ddot{\theta}_3 + d_1 = 0 \\
&c_{21}\ddot{\theta}_1 + c_{22}\ddot{\theta}_2 + c_{23}\ddot{\theta}_3 + d_2 = 0 \\
&c_{31}\ddot{\theta}_1 + c_{32}\ddot{\theta}_2 + c_{33}\ddot{\theta}_3 + d_3 = 0
\end{aligned}
\tag{4.22}
$$

Where all the constants are defined as

$$
\begin{aligned}
&c_{11} = (m_1 + m_2 + m_3)l_1^2; \\
&c_{12} = m_2 l_1 l_2 + m_3 l_1 l_2 \\
&c_{13} = m_3 l_1 l_3 \\
&d_1 = (m_1 + m_2 + m_3) g l_1 \theta_1 \\
&c_{21} = m_2 l_1 l_2 + m_3 l_1 l_2; \\
&c_{22} = (m_2 + m_3) l_2^2 \\
&c_{23} = m_3 l_2 l_3 \\
&d_2 = (m_2 + m_3) g l_2 \theta_2 \\
&c_{31} = m_3 l_1 l_3 \\
&c_{32} = m_3 l_2 l_3 \\
&c_{33} = m_3 l_3^2 \\
&d_3 = m_3 g l_3 \theta_3
\end{aligned}
$$

It was assumed that $m_1 = m_2 = m_3 = m$ and $l_1 = l_2 = l_3 = l$, linearizing the equations for the TP under vertical condition i.e., $\theta_1 = \theta_2 = \theta_3$ and θ is small, then Eqn. (4.22) converts in to Eqn. (4.23)

$$
\begin{aligned}
&3\ddot{\theta}_1 + 2\ddot{\theta}_2 + \ddot{\theta}_3 - 3gl^{-1}\theta_1 = 0 \\
&2\ddot{\theta}_1 + 2\ddot{\theta}_2 + \ddot{\theta}_3 - 2gl^{-1}\theta_2 = 0 \\
&\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 - gl^{-1}\theta_3 = 0
\end{aligned}
\tag{4.23}
$$

The Eqn. (4.23) can be represented in the following form

$$[M]\{\ddot{\theta}\}+[K]\{\theta\}=\{0\} \tag{4.24}$$

Where $[M]$ and $[K]$ are defined as

$$[M]=ml^2\begin{bmatrix}3&2&1\\2&2&1\\1&1&1\end{bmatrix},[K]=mgl\begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix} \tag{4.25}$$

The eigenvalues of the equation (4.24) can be calculated as

$$\left|-\lambda[M]+[K]\right|=m\begin{bmatrix}3g-3l\lambda & -2l\lambda & -l\lambda\\ -2l\lambda & 2g-2l\lambda & -l\lambda\\ -l\lambda & -l\lambda & g-l\lambda\end{bmatrix}=0 \tag{4.26}$$

That is

$$-l^3\lambda^3+9gl^2\lambda^2-18g^2l\lambda+6g^3=0,$$

Hence the eigenvalue will be

$$\begin{aligned}\lambda_1&=(3-\sqrt{3}\cos\alpha-3\sin\alpha)\frac{g}{l}\approx 0.4158\tfrac{g}{l},\\ \lambda_2&=(3-\sqrt{3}\cos\alpha+3\sin\alpha)\frac{g}{l}\approx 2.2943\tfrac{g}{l},\\ \lambda_3&=(3+2\sqrt{3}\cos\alpha)\frac{g}{l}\approx 6.2899\tfrac{g}{l},\end{aligned} \tag{4.27}$$

Where $\alpha=\dfrac{\arctan\sqrt{2}}{3}\approx 0.3184$.

All the above equations are independent of inertia of the system. It may be verified that frequency estimated by Eigenvalue analysis matches with the frequency obtained using FFT. Moreover, amplitude also matches with linearized value. For stance natural frequencies for the TP is $\omega_{01} = 0.4158\, g/l$, $\omega_{02} = 2.2943\, g/l$, $\omega_{03} = 6.2899\, g/l$. These results are consistent with the frequencies obtained using FFT.

4.4 MODELLING OF TRIPLE PENDULUM FOR DISTRIBUTED MASS AND LENGTH

If C1, C2 and C3 are the mid points of the lengths l_1 , l_2 and l_3 of distributed pendulum with mass m_1 , m_2 and m_3 respectively. If θ_1 is the angle from the first link, θ_2 is the angle from the second link and θ_3 is the angle from the third link.

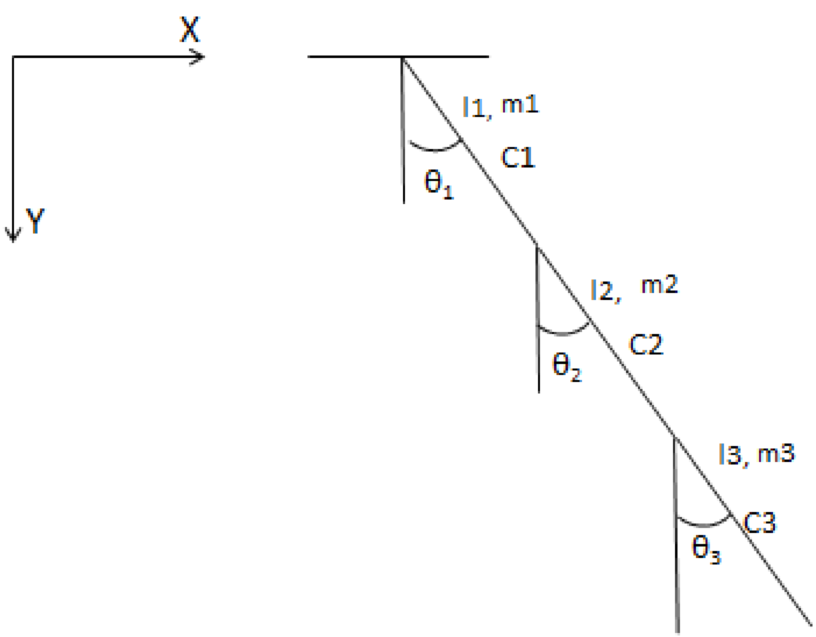


Fig 4.2 (a) Fig. 4.2(b)

Fig4.2-Triple pendulum with distributed mass

The coordinate value of C_1, C_2 and C3 will be described by the Eqn. (4.28)

$$
\begin{aligned}
&C1 = (\frac{l_1}{2}\sin\theta_1, -\frac{l_1}{2}\cos\theta_1),\\
&C2 = (l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2, -l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2)\\
&C3 = (l_1\sin\theta_1 + l_2\sin\theta_2 + \frac{l_3}{2}\sin\theta_3, -l_1\cos\theta_1 - l_2\cos\theta_2 - \frac{l_3}{2}\cos\theta_3)
\end{aligned}
\tag{4.28}
$$

The Kinetic Energy of the TP

$$
K.E. = \sum_{i=1}^{3}\frac{I_{ci}\dot{\theta}_i^{\,2}}{2} + \frac{m_i v_i^{\,2}}{2}
\tag{4.29}
$$

And the Potential energy of the system will be

$$
\begin{aligned}
&P.E. = -\frac{m_1 g l_1\cos\theta_1}{2} - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2/2) -\\
&m_3 g(l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3/2) +\\
&\frac{m_1 g l_1}{2} + m_2 g(l_1 + \frac{l_2}{2}) + m_3 g(l_1 + l_2 + \frac{l_3}{2})
\end{aligned}
\tag{4.30}
$$

The Lagrange of the system is defined using the Eqn. (4.29) and Eqn. (4.30)

$$
L = K.E. - P.E.
$$

$$
\begin{aligned}
&L = \frac{I_{c1}\dot{\theta}_1^2}{2} + \frac{m_1 l_1^2\dot{\theta}_1^2}{8} + \frac{I_{c2}\dot{\theta}_2^2}{2} + \frac{m_2}{2}\{l_1^2\dot{\theta}_1^2 + \frac{l_2^2\dot{\theta}_2^2}{4} + l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\}\\
&\frac{I_{c3}\dot{\theta}_3^2}{2} + \frac{m_3}{2}\{l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + \frac{l_3^2\dot{\theta}_3^2}{4} + 2l_1 l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 +\\
&l_1 l_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_3 - \theta_1) + l_2 l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2 - \theta_3)\} + \frac{m_1 g l_1\cos\theta_1}{2} +\\
&m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2/2) + m_3 g(l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_2/2)\\
&-\frac{m_1 g l_1}{2} - m_2 g(l_1 + \frac{l_2}{2}) - m_3 g(l_1 + l_2 + \frac{l_3}{2})
\end{aligned}
\tag{4.31}
$$

Where $I_{ci} = \dfrac{m_i l_i^2}{12}$, I is the moment of inertia of the system.

The Lagrange equation of motion can be obtained using the following formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} + \frac{\partial Q}{\partial \theta_i} = 0 \qquad i = 1, 2, 3 \tag{4.32}$$

Here Q is Rayleigh Damping.

$$Q = \frac{c_1}{2}\dot{\theta}_1^{\,2} + \frac{c_2}{2}(\dot{\theta}_1 - \dot{\theta}_2)^2 + \frac{c_3}{2}(\dot{\theta}_2 - \dot{\theta}_3)^2 \tag{4.33}$$

It may be noted that $c1$, $c2$ and $c3$ are the constants related with frictional damping of top, middle and bottom pivots about which the three pendula are rotating.

The matrix of the following form can be obtained by combining the $\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3$ and constants

$$\begin{aligned}
C_{11}\ddot{\theta}_1 + C_{12}\ddot{\theta}_2 + C_{13}\ddot{\theta}_3 + d_1 &= 0 \\
C_{21}\ddot{\theta}_1 + C_{22}\ddot{\theta}_2 + C_{23}\ddot{\theta}_3 + d_2 &= 0 \\
C_{31}\ddot{\theta}_1 + C_{32}\ddot{\theta}_2 + C_{33}\ddot{\theta}_3 + d_3 &= 0
\end{aligned} \tag{4.34}$$

The coefficients $C_{11}, C_{12}, C_{13}, d_1, C_{21}, C_{22}, C_{23}, d_2, C_{31}, C_{32}, C_{33}, d_3$ are defined in the Appendix A.

4.5 LINEAR STABILITY ANALYSIS OF TRIPLE DISTRIBUTED PENDULUM

The Eqn. (4.30) can be represented in the following form

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\} \tag{4.35}$$

Assuming that mass and length are equal so mass $m_1 = m_2 = m_3 = m$ and length $l_1 = l_2 = l_3 = l$, same method is used as applied in section (4.3).

Where $[M]$ and $[K]$ are defined as

$$[M] = \frac{ml^2}{2}\begin{bmatrix} I_c + 5 & 3 & 1 \\ 3 & I_C + 5/2 & 1 \\ 1 & 1 & I_c + 1/2 \end{bmatrix}, [K] = \frac{mgl}{2}\begin{bmatrix} 5\,0\,0 \\ 0\,3\,0 \\ 0\,0\,1 \end{bmatrix} \tag{4.36}$$

The eigenvalues of the equation (4.25) can be calculated as

$\left| -\lambda[M] + [K] \right| = 0$, and the eigenvalues of the system will be

$$\begin{aligned} \lambda_1 &= 4.75714 \\ \lambda_2 &= 29.43 \\ \lambda_3 &= 105.039 \end{aligned} \tag{4.37}$$

The magnitude of eigenvalues will be higher in case of distributed system comparing to the lumped system.

4.6 MODELLING OF TRIPLE LINK INVERTED PENDULUM ON CART

The TLIP shown in Fig. 4.3 shows that M is Mass of the cart, m_i - Mass of the i^{th} link, l_i - distance from the lower position sensor to the center of gravity of the i^{th} link, L_i - Total length of the i^{th} link, I_i - Mass moment of inertia of the i^{th} link about its center of gravity, r - Cart's position from the middle of the rail track, θ_i - Angle of the i^{th} link from the vertical position, C_c Dynamic friction coefficient between the cart and the track, C_i - Dynamic friction coefficient for the i^{th} link, μ_j - Coulomb friction coefficient for the j^{th} link.

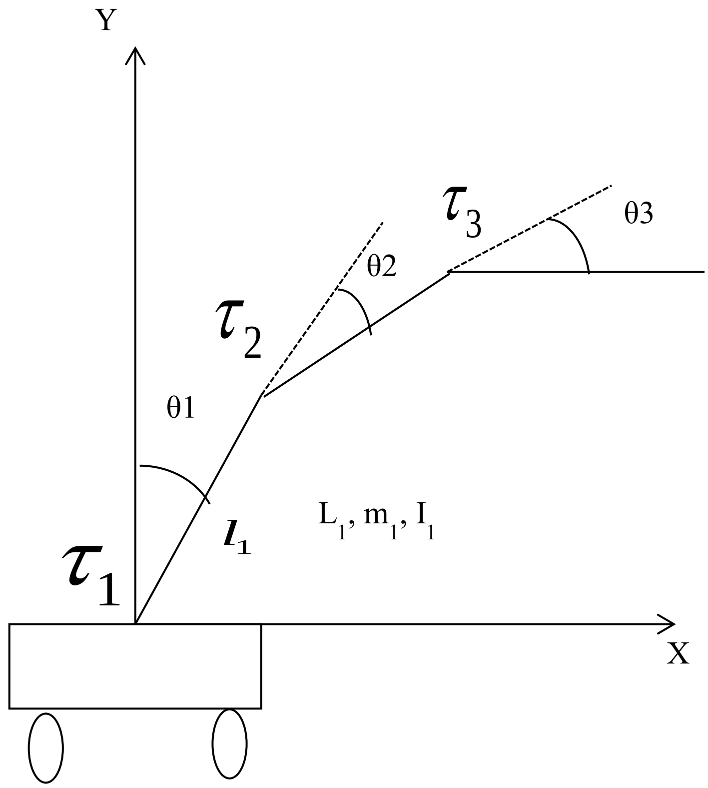


Fig. 4.3 Triple link Inverted pendulum system on cart

The system nonlinear dynamics equations can be given in the following form

$$F(q)\ddot{q} = -G(q,\dot{q})\dot{q} - H(q) + L(q,u) \tag{4.38}$$

Where

$$F(q) = \begin{bmatrix} A_1 & A_2\cos(\theta_1) & A_3\cos(\theta_2) & A_4\cos(\theta_3) \\ A_9\cos(\theta_1) & A_{10} & A_{11}\cos(\theta_1\text{-}\theta_2) & A_{12}\cos(\theta_1\text{-}\theta_3) \\ A_{18}\cos(\theta_2) & A_{19}\cos(\theta_1\text{-}\theta_2) & A_{20} & A_{21}\cos(\theta_2\text{-}\theta_3) \\ A_{28}\cos(\theta_3) & A_{29}\cos(\theta_1\text{-}\theta_3) & A_{30}\cos(\theta_2\text{-}\theta_3) & A_{31} \end{bmatrix} \tag{4.39}$$

$$G(q,\dot{q})=\begin{bmatrix} A_5 & A_6\sin(\theta_1)\dot{\theta}_1 & A_7\sin(\theta_2)\dot{\theta}_2 & A_8\sin(\theta_3)\dot{\theta}_3 \\ 0 & A_{13} & A_{14}\sin(\theta_1-\theta_2)\dot{\theta}_2+A_{15} & A_{16}\sin(\theta_1-\theta_3)\dot{\theta}_3 \\ 0 & A_{22}\sin(\theta_1-\theta_2)\dot{\theta}_1+A_{23} & A_{24} & A_{25}\sin(\theta_2-\theta_3)\dot{\theta}_3+A_{26} \\ 0 & A_{33}\sin(\theta_1-\theta_3)\dot{\theta}_1 & A_{35}\sin(\theta_2-\theta_3)\dot{\theta}_2+A_{36} & A_{32} \end{bmatrix} \tag{4.40}$$

$$q=\begin{bmatrix} r \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix},\quad H(q)=\begin{bmatrix} 0 \\ A_{17}\sin(\theta_1) \\ A_{27}\sin(\theta_2) \\ A_{34}\sin(\theta_3) \end{bmatrix},\quad L(q,u)=\begin{bmatrix} K_s u - sign(r)\mu_r N_r \\ -sign(\theta_1)\mu_{\theta 1}N_{\theta 1} \\ -sign(\theta_1)\mu_{\theta 2}N_{\theta 2} \\ -sign(\theta_1)\mu_{\theta 3}N_{\theta 3} \end{bmatrix} \tag{4.41}$$

Where u is the control input and K_s is the overall systems input. The system parameters, as given in the appendix matrix A, are determined either by the direct measurements or from the experimental data [33].

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -7.6 & -0.156 & -0.0005 & -4.9 & 0.0005 & -0.0005 & 0 \\ 0 & 38.9 & -23.9 & -0.07 & 11.11 & -0.0046 & 0.0087 & -0.0037 \\ 0 & -37.03 & 82.75 & -2.01 & -10.55 & 0.0087 & -0.0234 & 0.0253 \\ 0 & -1.7 & -52.8 & 71.99 & -0.49 & -0.0037 & 0.0253 & -0.4 \end{bmatrix} \tag{4.42}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.903 \\ -2.02 \\ 1.91 \\ 0.09 \end{bmatrix} \tag{4.43}$$

The A_i is the systems constants given in the appendix B.

4.6 CHAPTER SUMMARY

The chapter describes the modelling of triple pendulum using Euler-Lagrange and momenta based approach. Modelling of triple Pendulum is obtained for lumped and distributed system both and verified from both the technique. Modelling is written finally in matrix form. From the matrix form also the linear stability analysis has been done. Modelling of triple link inverted pendulum on cart is also discussed in the chapter.

CHAPTER 5

RESULTS AND DISCUSSION

The chapter explains the MATLAB simulation and experimental observations. It presents the mass and length dependent behaviour of double pendulum. The effect of damping on the dynamics of double and triple pendulum is discussed. Double and triple pendulum shows chaotic behaviour so all the tools for the measurement of chaos like Time series analysis, FFT, Poincare and Lyapunov exponent is also shown which validate the experimental setup. In the dynamics of DP and TP angles are started from 10 degree and goes to 90 degree and same phenomenon are using in experimental setup. The optimal control technique is used for control of triple link inverted pendulum. The simulation is carried out on MATLAB 12 .0 in mathematical window editor. The differential equations are simulated numerically in MATLAB using ode 45 solver.

5.1 MASS AND LENGTH DEPENDENT BEHAVIOUR OF DOUBLE PENDULUM

The differential Eqn. (3.23) & (3.24) are simulated numerically in MATLAB using ode 45. All simulations are carried out for a fixed initial condition i.e. $(\theta_1, \theta_2, p_1, p_2)$ = $(0.2, 0.2828, 0, 0)$ for mass and length effect. To understand the mass and length chaotic behaviour of double pendulum time series of angular velocity of the two pendulums, Poincare map and the average value of Lyapunov exponent are plotted.

Fig. 5.1 presents the plots concerning the time series (angular velocity), Poincare map and Lyapunov exponent for $m_1 = 1.0\ \mathrm{kg}$, $m_2 = 1.0\ \mathrm{kg}$, $l_1 = 1.0\ \mathrm{m}$ and $l_2 = 1.0\ \mathrm{m}$, the Hamiltonian energy $E = -28.62\mathrm{J}$. It is obvious from the plot Fig 5.1 (a) that angular velocity of both pendula is periodic and both are in phase. Moreover, corresponding Poincare map is also regular as evident in plot Fig 5.1 (b) of the same Fig 5.1. Also the average Lyapunov exponent e is found to be -0.1607 which is negative. As mentioned earlier, dynamical system with negative exponent does not show the chaotic motion.

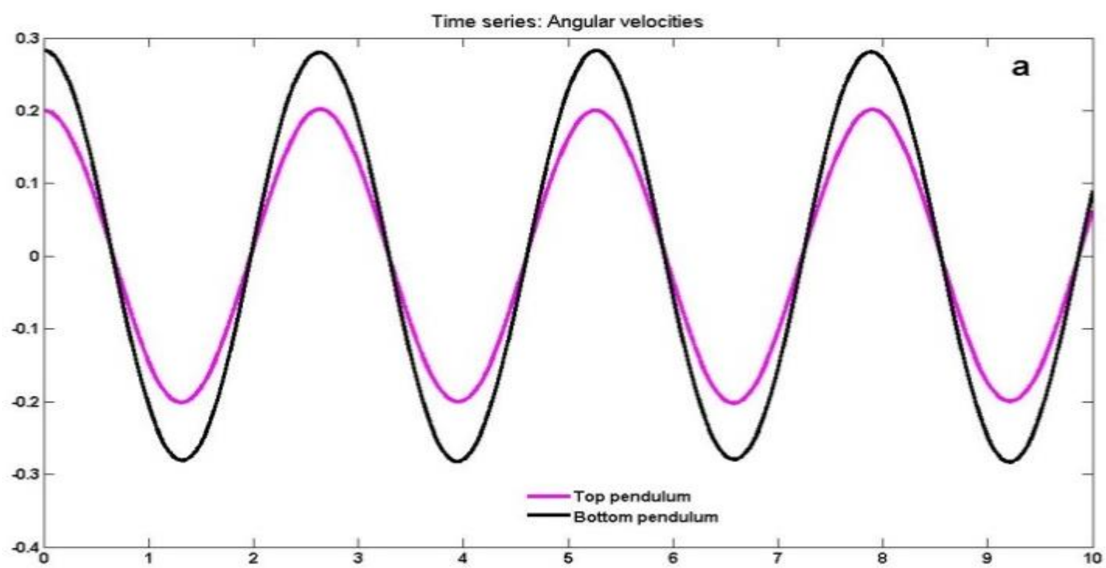


Fig 5.1 (a) Time series analysis

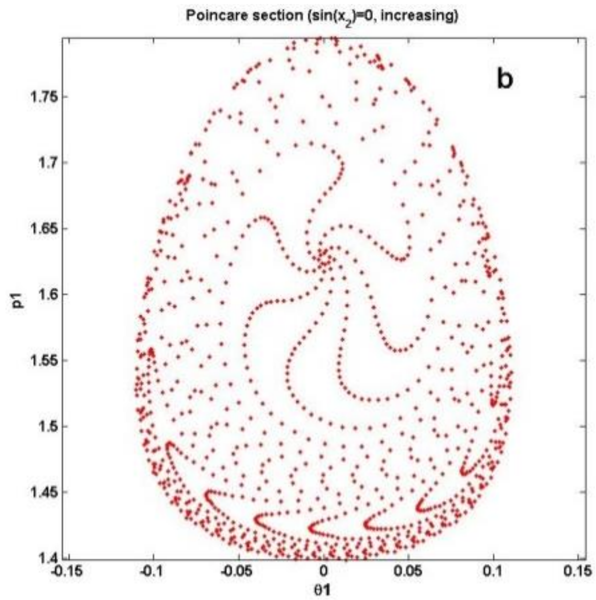


Fig. 5.1(b) Poincae Map

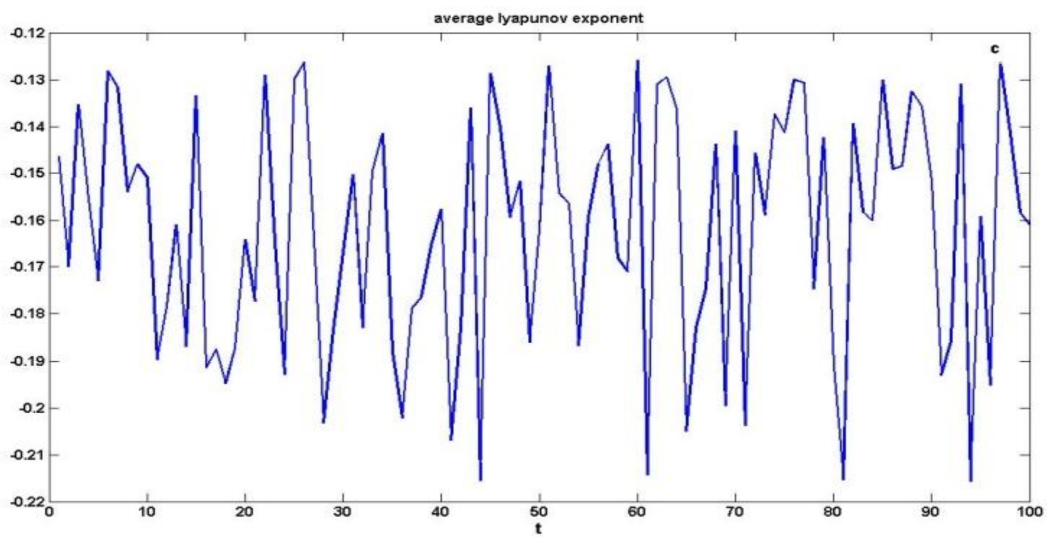


Fig. 5.1(c) Lyapunov Exponent

Fig. 5.1 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1, m_2 = 1$, $l_1 = 1$ and $l_2 = 1$, Hamiltonian $E = -28.62$.

If the mass of the top pendulum is increased to $m_1 = 5$ while other parameters were considered the same as earlier, now the energy corresponding to Hamiltonian increased to $E = -67.04$. Time series plot of angular velocity of Fig 5.2 (a) show that the motion is now quasi-periodic as there are two time period in the motion. And the corresponding Poincare map shows to be regular but with large area of null. The average Lyapunov exponent $e = -0.1086$ is negative. This value of e is larger than in Fig.5.2(c). Thus it may be concluded that chaotic tendency of the DP increased with mass of the top pendulum.

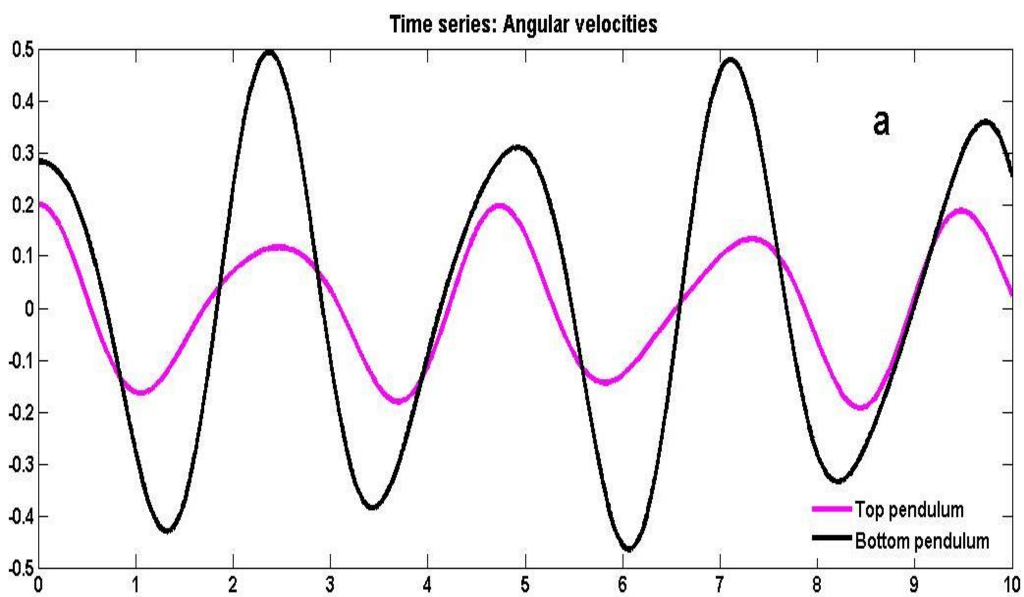


Fig 5.2 (a) Time series analysis

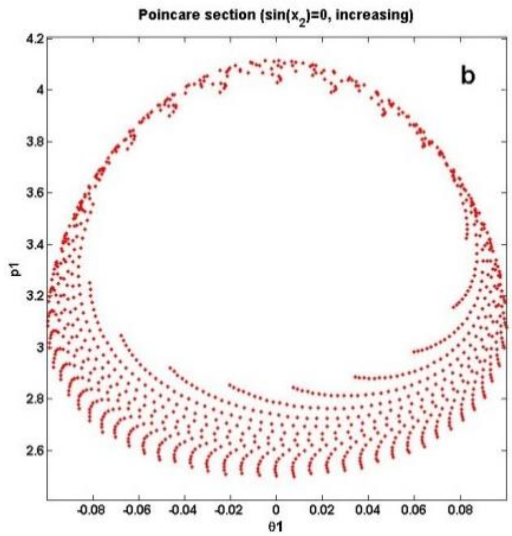


Fig. 5.2 (b) Poincae Map

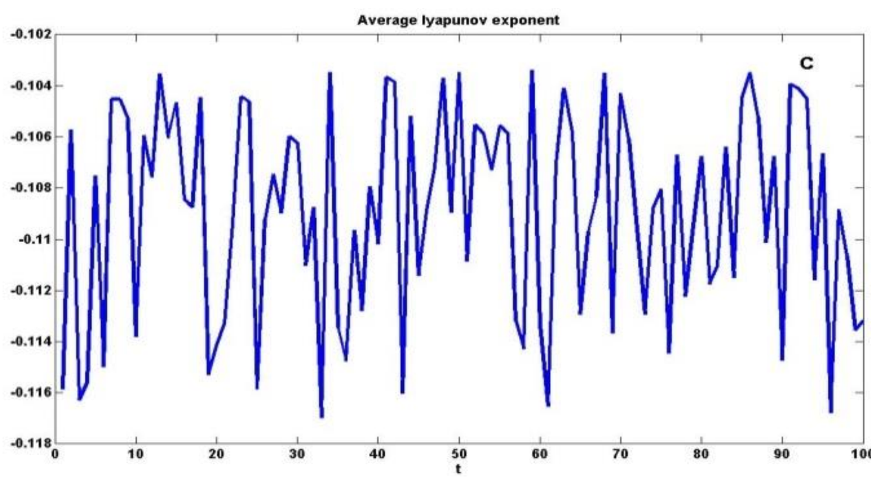


Fig. 5.2(c) Lyapunov Exponent

Fig.5.2 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for

$m_1 = 5$, $m_2 = 1$, $l_1 = 1$ and $l_2 = 1$, Hamiltonian $E = -67.04$.

Now in case of change in magnitude of the mass of the bottom pendulum i.e., $m_2 = 5$, the time series Fig.5.3 (a) shows that the top pendulum is showing period oscillation while the bottom pendulum is random. The chaotic behavior is also clear in Fig. 5.3 (b) where Poincare not in regular in shape it is clearly irregular in nature. The value of average Lyapunov exponent is $e = 0.0399$. As noted earlier that positive value of the average Lyapunov exponent results in chaotic oscillations. It may be concluded from fig. 5.3 that the bottom makes the DP more chaotic than the top pendulum.

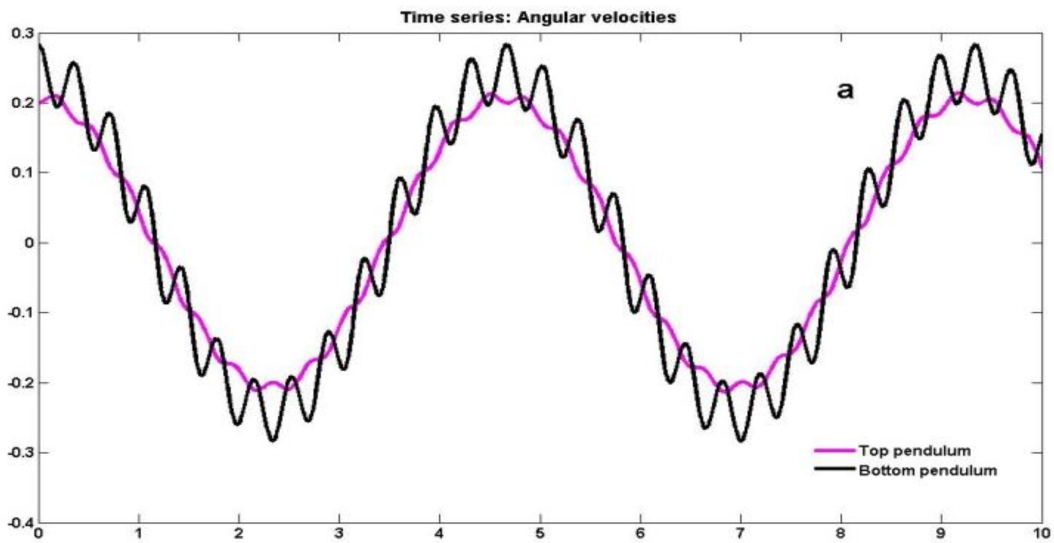


Fig 5.3 (a) Time series analysis

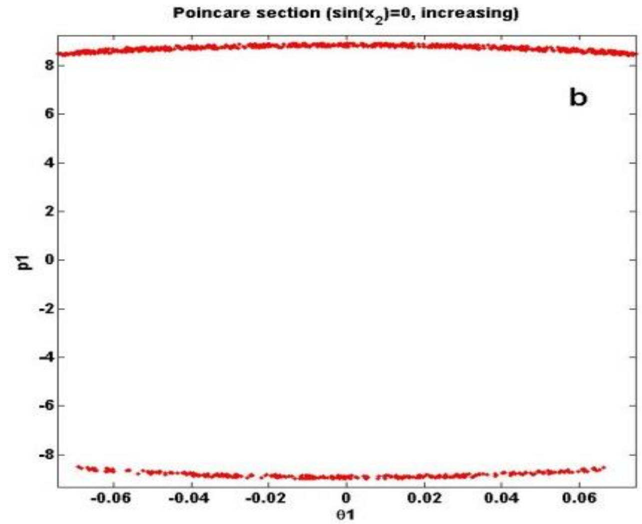


Fig. 5.3 (b) Poincae Map

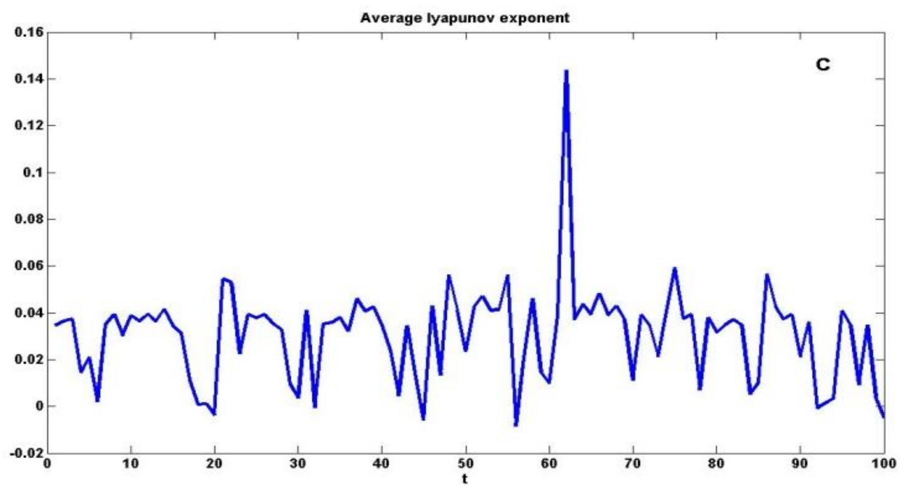


Fig. 5.3 (c) Lyapunov Exponent

Fig. 5.3 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for

$m_1 = 1, m_2 = 5$, $l_1 = 1$ and $l_2 = 1$ for Hamiltonian $E = -104.68$.

After seeing the effect of mass, it becomes interesting to know how arm length of pendula affects its chaotic dynamics. Length of the top pendulum was changed from $l_1 = 1$ to $l_1 = 3$. The result from the time series in Fig. 5.4(a) shows that the bottom pendulum is showing quasi-periodic oscillation while the top pendulum is still periodic. The value of average Lyapunov exponent is equal to -0.0991. It is concluded that increasing arm length of the top pendulum results in quasi-periodic motion.

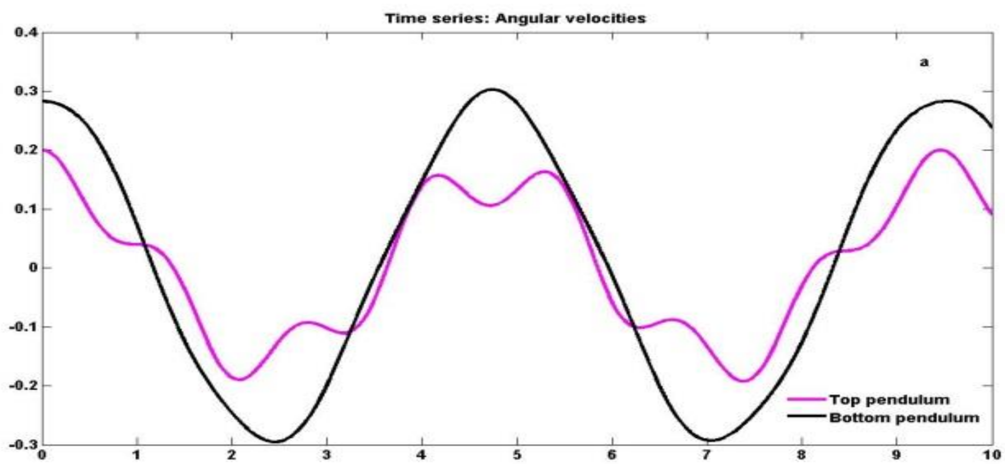


Fig 5.4 (a) Time series analysis

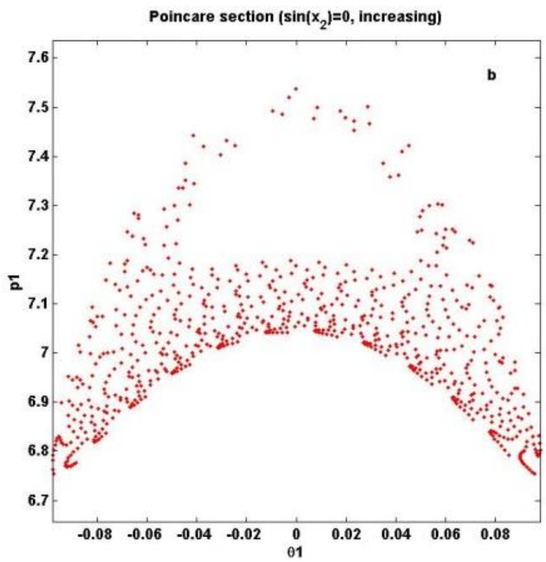


Fig. 5.4 (b) Poincae Map

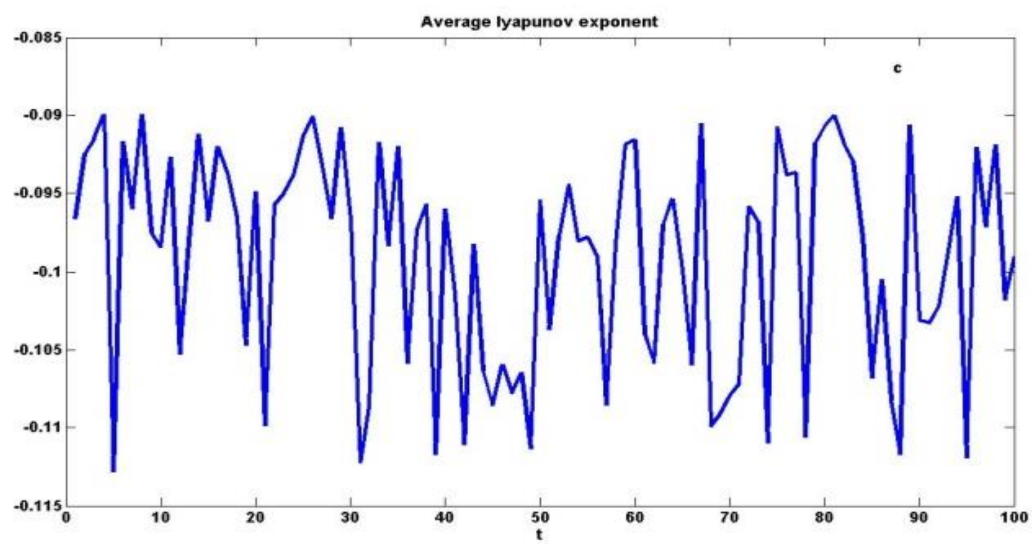


Fig. 5.4 (c) Lyapunov Exponent

Fig.5.4 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1$, $m_2 = 1$, $l_1 = 3$ and $l_2 = 1$, Hamiltonian $E = -67.04$.

Finally, arm length of the bottom pendulum was also changed in the simulation for $l_2 = 3$. The value of average Lyapunov exponent is $e = 0.0234$ .The results in Fig. (5.5) shows that DP is ultimately moving towards the chaos.

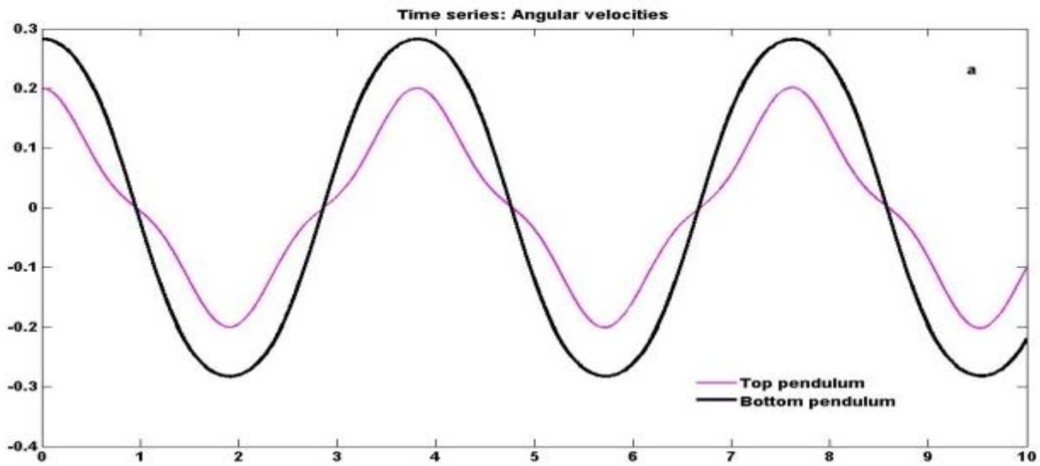


Fig 5.5 (a) Time series analysis

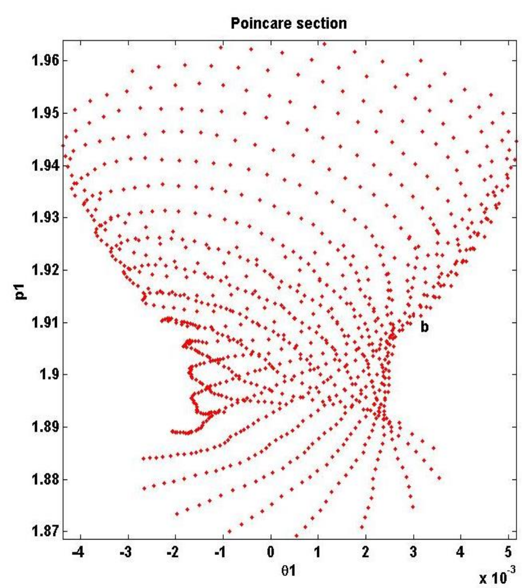


Fig. 5.5 (b) Poincare Map

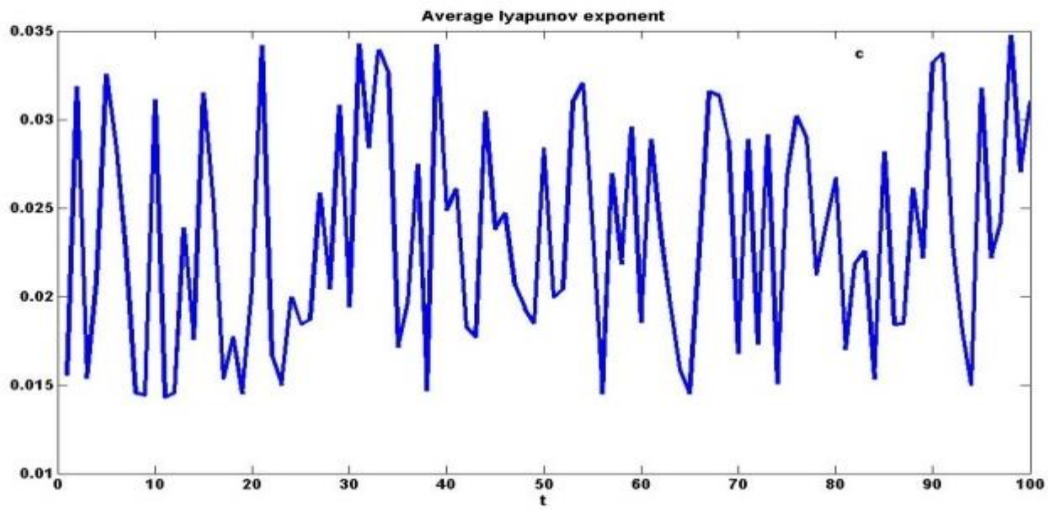


Fig. 5.5 (c) Lyapunov Exponent

Fig.5.5 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1, m_2 = 1, l_1 = 1$ and $l_2 = 3, E = -47.44J$.

The summary of the above simulations is presented in the Table 5.1.

TABLE 5.1 (Summary of mass and length dependent behaviour of DP)

Mass (Kg) and length (m) of each pendulum	Lyapunov exponent e and Hamiltonian energy E(Joule)	Chaotic/non-chaotic behavior
$m_1 = 1, m_2 = 1, l_1 = 1$ $l_2 = 1$	$e = -0.1607$ $E = -28.62J$	Periodic
$m_1 = 5, m_2 = 1, l_1 = 1$ $l_2 = 1$	$e = -0.1086$ $E = -67.04J$	Quasi-periodic
$m_1 = 1, m_2 = 5, l_1 = 1$ $l_2 = 1$	$e = 0.0399$ $E = -104.68J$	Chaos
$m_1 = 1, m_2 = 1, l_1 = 3$ $l_2 = 1$	$e = -0.0991$ $E = -67.04J$	Quasi-periodic
$m_1 = 1, m_2 = 1, l_1 = 1$ $l_2 = 3$	$e = 0.0234$ $E = -47.44J$	Chaos

Thus it can be concluded that there is an optimal combination of mass and arm length of the DP at which chaotic tendency of the system will be minimum and maximum.

5.2 TIME SRIES OF DOUBLE PENDULA AND TRIPLE PENDULA

The time series plot of Fig.5.6 (a) shows that for a small change (10^{-9}) in initial conditions $(\pi/2, 0, \pi/2, 0)$ and $(\pi/2 + 10^{-9}, 0, \pi/2, 0)$.It is clear from the fig 5.6 (a) that initially both the trajectories are same but after some time 12-13 second in case of top and bottom pendulum both the trajectories deviates from each other and that can be seen easily. In case top pendulum deviation is less where as in case of bottom pendulum deviation is more. Thus the bottom pendulum is more chaotic than the top pendulum. The change in sudden behaviour of nonlinear systems may give rise to the complex behaviour called Chaos and this phenomenon can be seen in fig. 5.6(a) where with the change of 10^{-9} in the initial condition after sometime around 12 -13 sec trajectories separates from each other. The time series analysis of triple pendulum is shown in fig. 5.6 (b). Hence the system can chaotic and this can be verified from the other tools for the measurement of chaos.

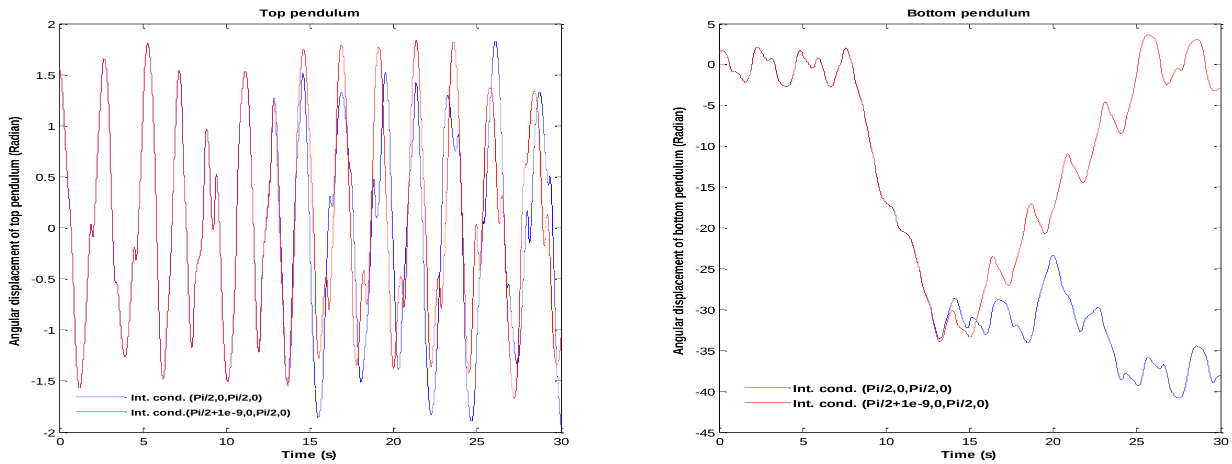


Fig.5.6(a) Time series plot for top and bottom double pendulum for two initial conditions.

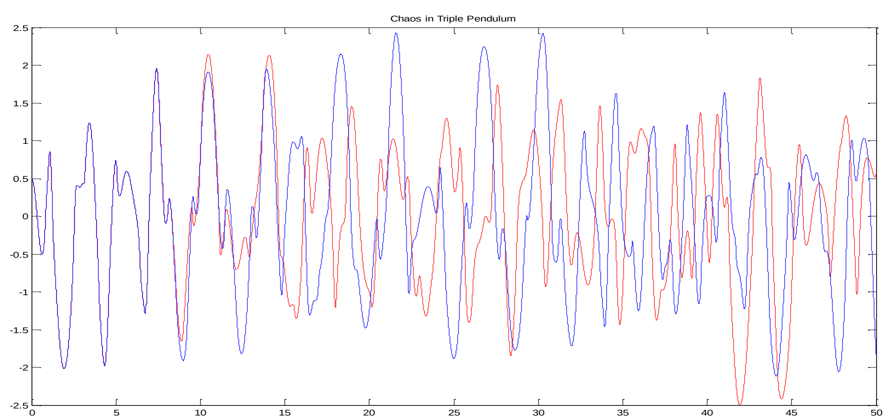


Fig. 5.6 (b) Time series for triple pendulum

Fig. 5.6 Time series analysis for double and triple Pendulum

5.3 FFT ANALYSIS

FFT analysis is one of the tool for the measurement of chaos so FFT analysis has been done for the Double and Triple pendulum for verifying the Chaotic behaviour.

5.3.1 FFT ANALYSIS OF DOUBLE PENDULUM

If a system shows periodic doubling [52] or tripling that is a sign of chaos that can be seen from the following figure 5.7. FFT analysis for DP with lumped mass system is shown in Fig. 5.7 (a) in which two frequencies can be seen and the values of these two frequencies are 0.36 and 0.88 Hertz respectively for top and bottom link.

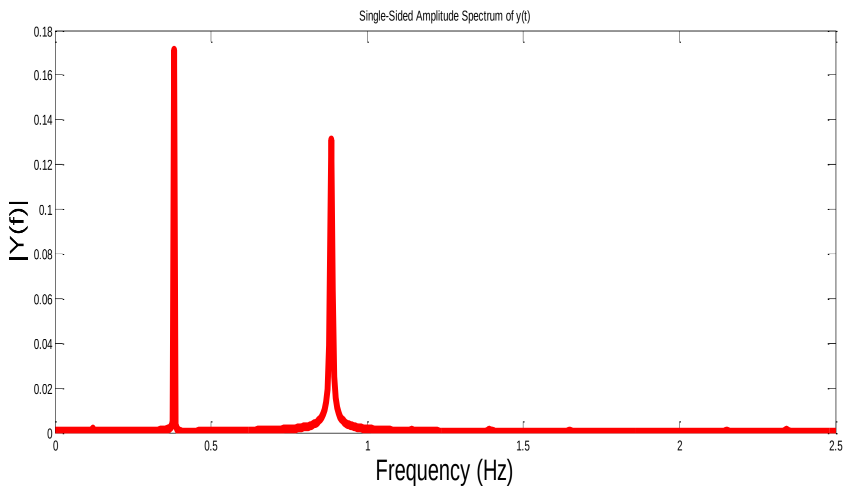


Fig. 5.7 (a) FFT Analysis of DP with lumped mass at $\pi/10$ Degree

If the angle increased to π /4 then in place of two frequency it shows many frequency (4 to 5 frequency) as can be seen from Fig. 5.7 (b).

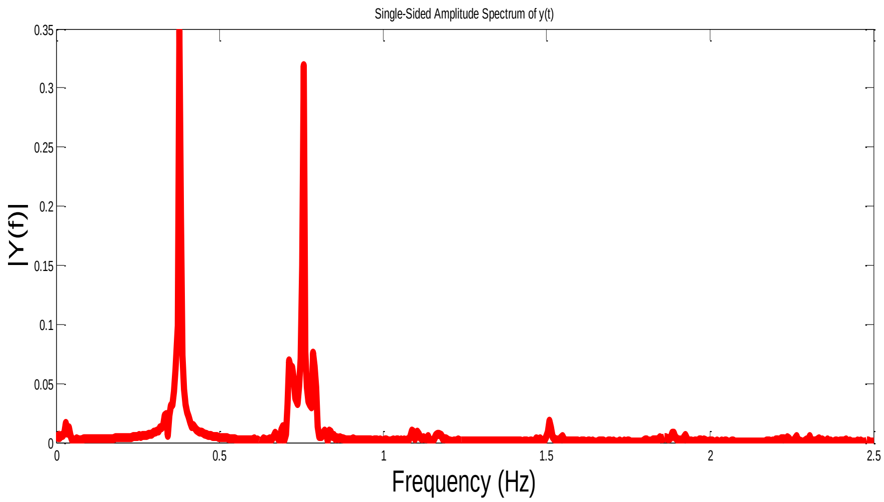


Fig. 5.7 (b) FFT analysis of DP with lumped mass at π /4 degree

If the angle increased further then multiple frequency can be seen from fig. 5.7 (c).

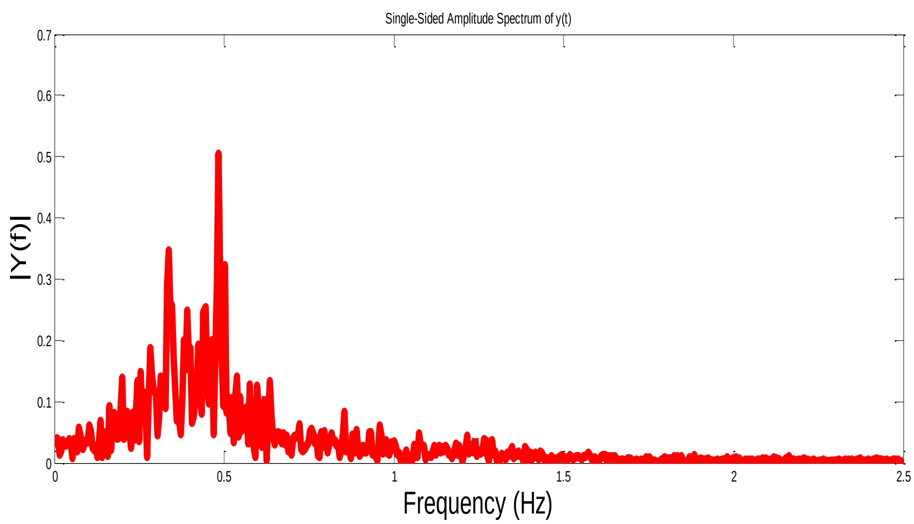


Fig. 5.7 (c) FFT analysis of DP with lumped mass at π /2 Degree

FFT analysis for DP with distributed mass system is shown in Fig. 5.8 (a) in which two frequencies can be seen and the values of these two frequencies are

0.38 and 0.92 Hertz respectively for top and bottom link which is higher from lumped system.

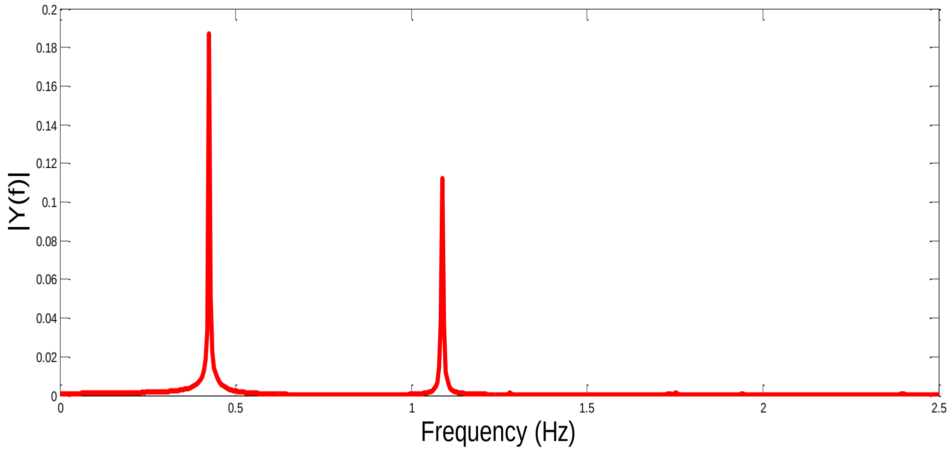


Fig. 5.8 (a) FFT analysis of DP with distributed mass at π /10 Degree

If the angle increased to π /4 then in place of two frequency it shows many frequency (5 to 6 frequency) as can be seen from fig. 5.8 (b).

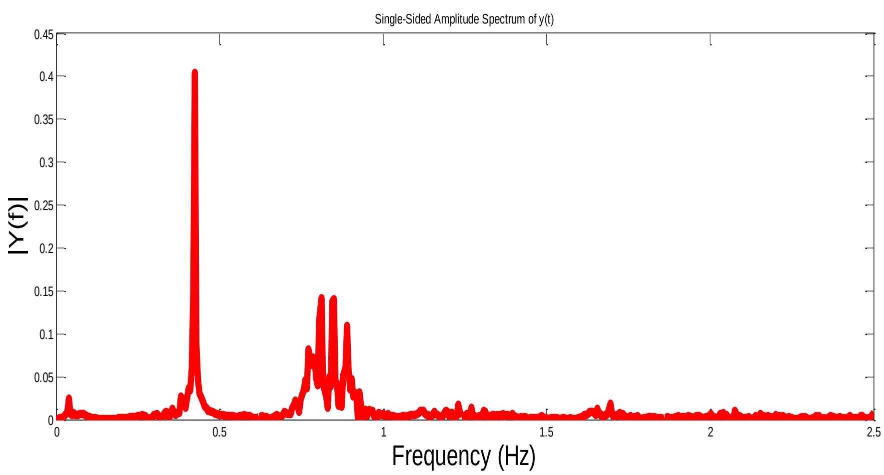


Fig. 5.8 (b) FFT analysis of DP with distributed mass at π /4 Degree

If the angle increased further to π /3 then multiple frequency can be seen from fig. 5.8 (b).

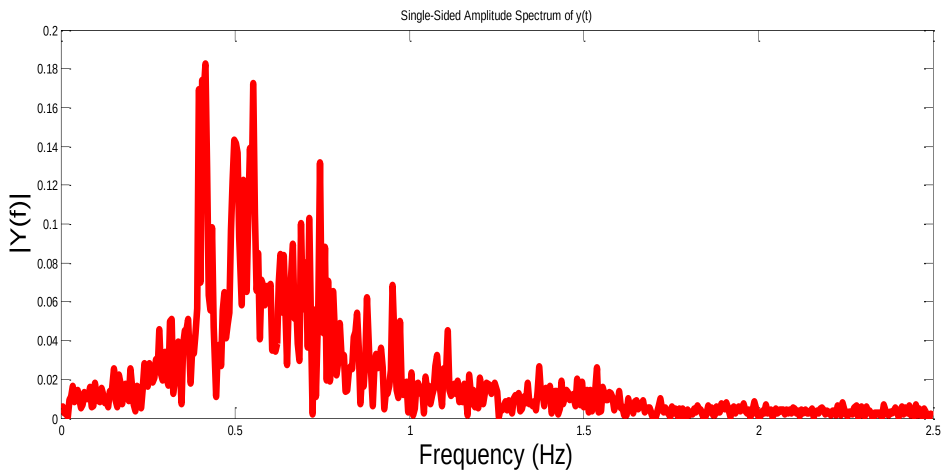


Fig. 5.8 (c) FFT analysis of DP with distributed mass at π /3 Degree

5.3.2 FFT ANALYSIS OF TRIPLE PENDULUM

FFT analysis for TP with lumped mass system is shown in Fig. 5.9 (a) in which three frequencies can be seen and the values of these three frequencies are 0.32, 0.75 and 1.25 Hertz respectively for top, middle and bottom link.

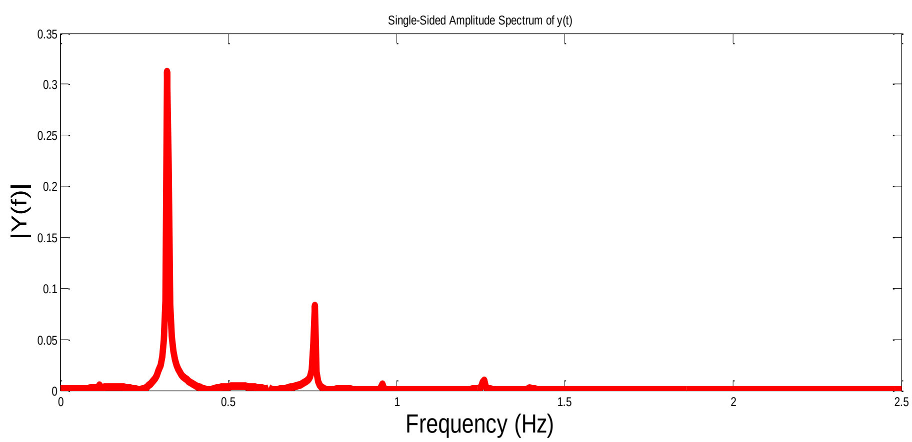


Fig. 5.9 (a) FFT analysis of TP with lumped mass at π /10 degree

If the angle increased to π /4 then in place of three frequency it shows many frequency (6-7 frequency) as can be seen from fig. 5.9 (b).

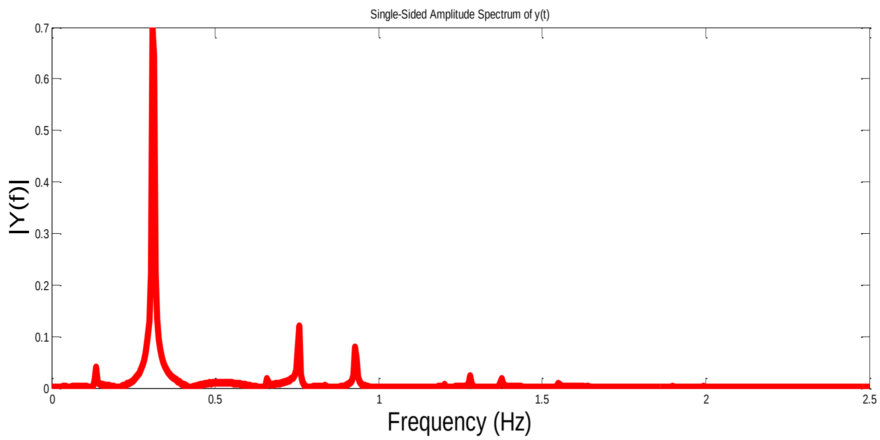


Fig. 5.9 (b) FFT analysis of TP with lumped mass at π /4 degree

If the angle increased further then multiple frequency can be seen from fig. 5.9 (c).

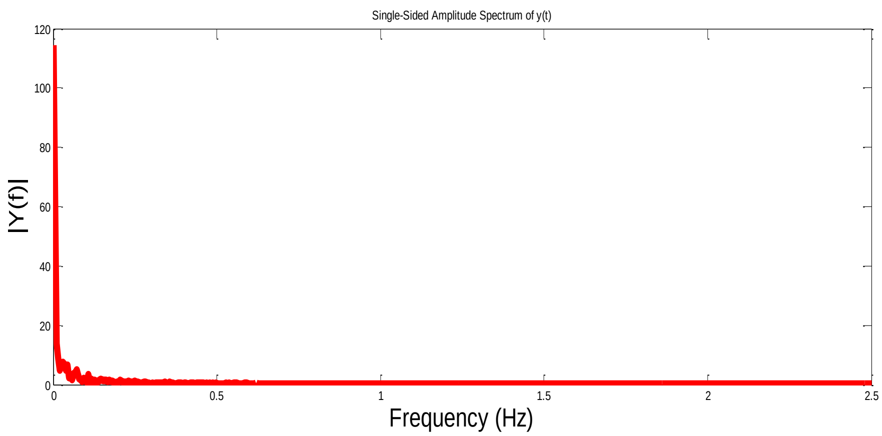


Fig. 5.9 (c) FFT analysis of TP with lumped mass at π /2 degree

FFT analysis for TP with distributed mass system is shown in Fig. 5.10 (a) in which three frequencies can be seen and the values of these three frequencies are 0.34, 0.86 and 1.63Hertz respectively for top, middle and bottom link and it is higher than triple pendulum with lumped system.

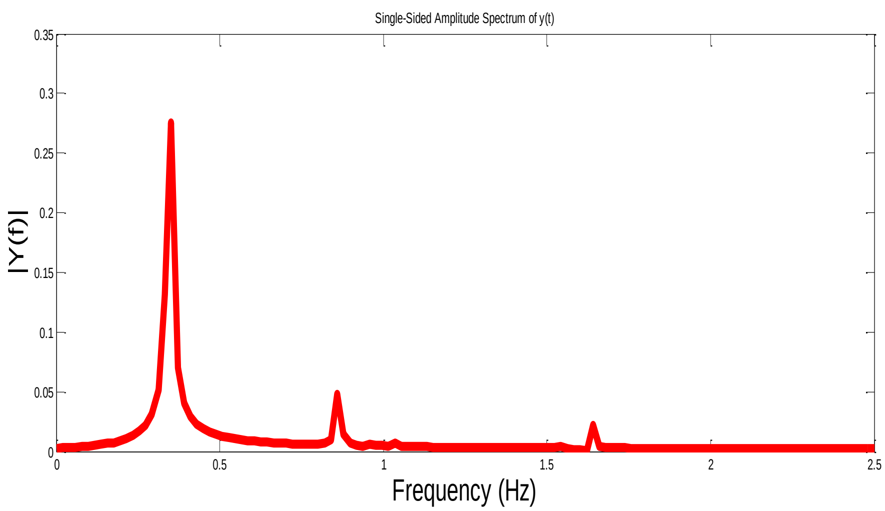


Fig. 5.10 (a) FFT analysis of TP with distributed mass at π /10 degree

If the angle increased to π /4 then in place of three frequency it shows many frequency (6-7 frequency) as can be seen from fig. 5.10 (b).

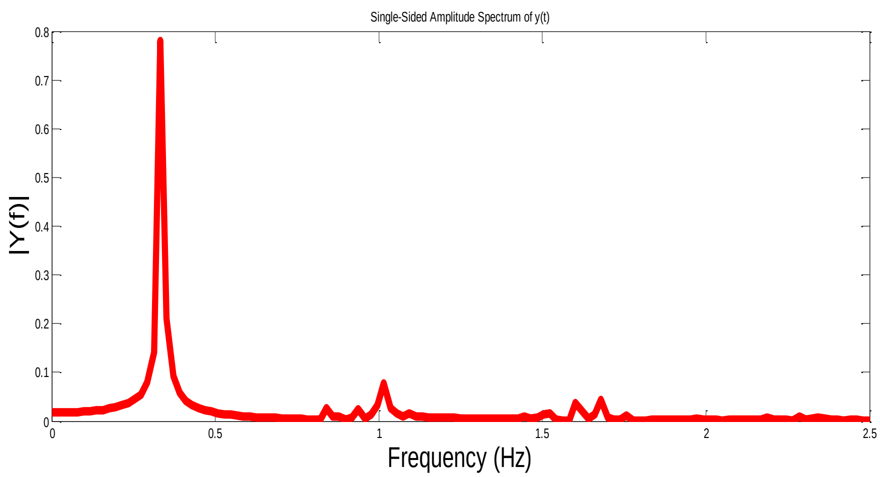


Fig. 5.10 (b) FFT analysis of TP with distributed mass at π /4 degree

If the angle increased further then multiple frequency can be seen from Fig. 5.10 (c).

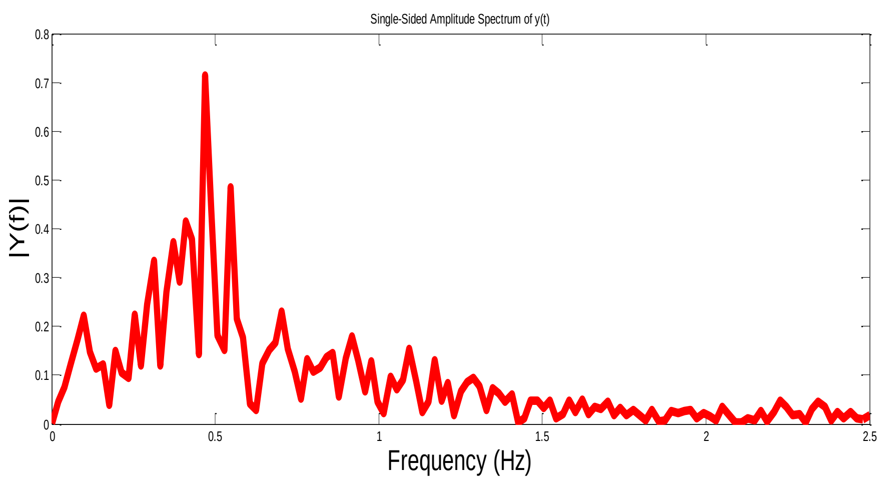


Fig. 5.10 (c) FFT Analysis of TP with Distributed Mass at π /2 Degree

So it can be seen clearly from the FFT analysis that with the increase in small angle chaotic tendency increases. Also going from lumped to distributed system frequency increases and it is maximum in case of bottom pendulum. Based on fig. 5.7 (a), 5.8 (a), 5.9 (a) and 5.10 (a) following table 5.2 can be made.

TABLE 5.2 FREQUENCY OF LUMPED AND DISTRIBUTED SYSTEM

Types of Pendulum	Lumped System Frequency	Distributed System Frequency
Simple Pendulum	0.49	0.61
Double Pendulum	0.36 0.88	0.38 0.92
Triple Pendulum	0.32 0.75 1.25	0.34 0.86 1.63

From the table it is clear that

1-Natural frequency of the linearized system increases from lumped to distributed system.

2- Order of the frequency: Top<Middle< Bottom Pendulum.

5.4 POINCARE ANALYSIS

The numerical analysis of the pendulum will involve the concept name Poincare sections, which are created as follows. Whenever a trajectory meets some "stopping condition," all variables (aside from that used to determine the condition) are plotted. This is repeated for qualitatively different trajectories. The resulting Poincare section is a map rather than a vector field; it has one dimension less than the phase space. Different Poincare has been obtained from the dynamic analysis of double and triple pendulum.

5.4.1 POINCARE OF DOUBLE PENDULUM

Fig. 5.11 (a), (b) shows the Poincare map for Double pendulum for lumped and distributed system at initial condition for $\pi/10$ degree. It is clear from the fig. 5.11 (a), (b) that although the Poincare is symmetrical but for distributed system it shows some irregularity.

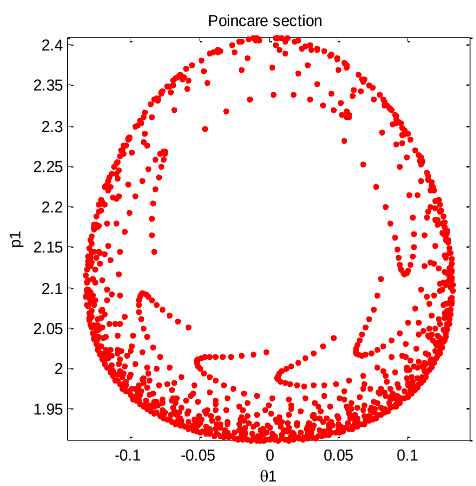


Fig. 5.11 (a)

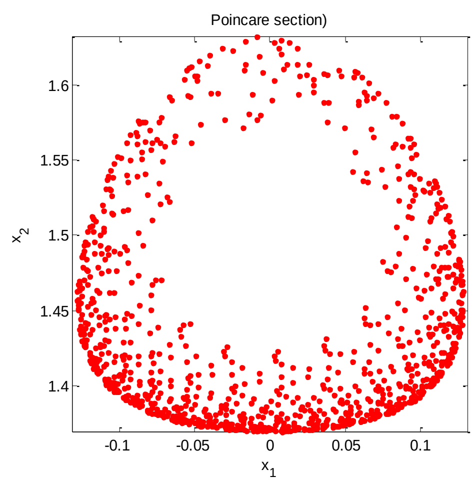


Fig. 5.11 (b)

Fig. 5.11 (a) and (b) DP with Lumped and Distributed system at starting point $y_0 = [\pi/10; \pi/10; 0; 0]$;

Fig. 5.12 (a), (b) shows the Poincare map for Double Pendulum for lumped and distributed system at initial condition for $\pi/4$ degree. It is clear from the fig. 5.12 (a),(b) that with the increase in small angle (going from $\pi/10$ to $\pi/4$) Poincare shows irregularity.

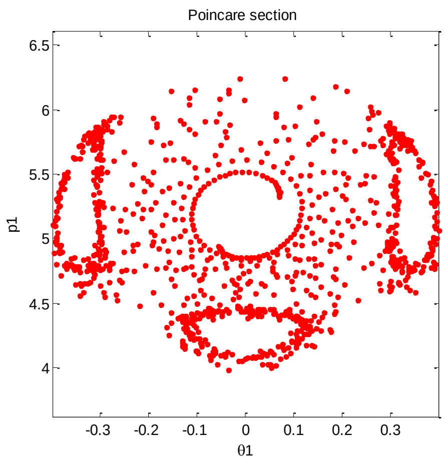


Fig. 5.12 (a)

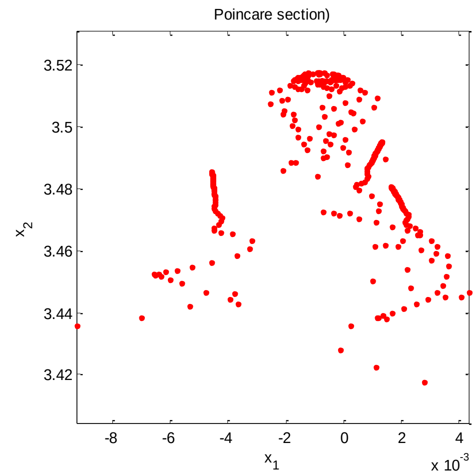


Fig 5.12 (b)

Fig. 5.12 (a) and (b) DP with lumped and distributed at $y_0 = [\pi /4; \pi /4; 0; 0]$;

Fig. 5.13 (a), (b) shows the Poincare map for Double pendulum for lumped and distributed system at initial condition for π /2 degree. It is clear from the fig. 5.13 (a),(b) that with the further increase in angle (going from π /4 to π /2) Poincare shows completely irregular shape.

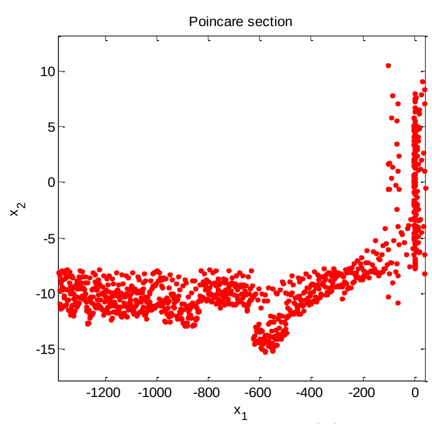


Fig. 5.13 (a)

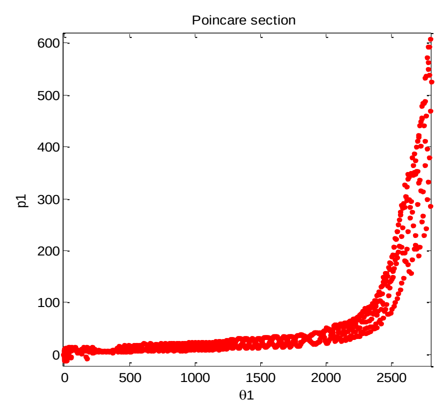


Fig 5.13 (b)

Fig. 5.13 (a) and (b) DP with lumped and distributed at $y_0 = [\pi /2; \pi /2; 0; 0]$;

5.4.2 POINCARE OF TRIPLE PENDULUM

Fig. 5.14 (a), (b) shows the Poincare map for Triple Pendulum for lumped and distributed system at initial condition for π /10 degree. It is clear from the figure 5.14 (a), (b) that for lumped system it shows some regular shape and for distributed system it is not regular one.

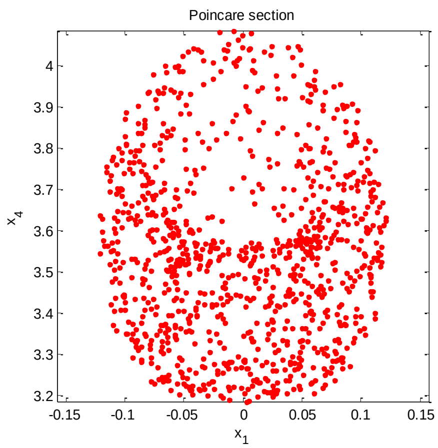


Fig. 5.14 (a)

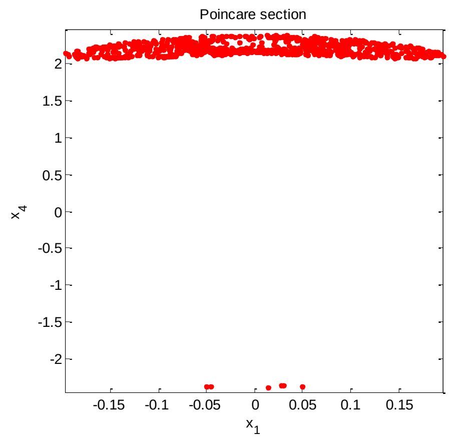
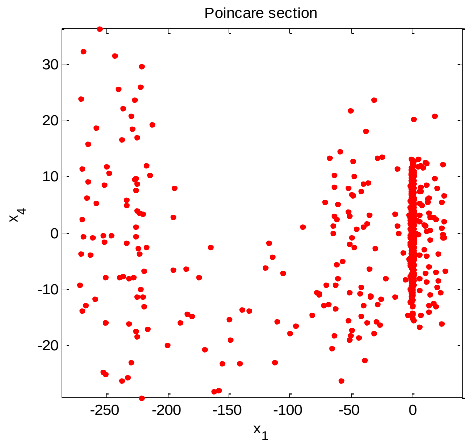


Fig 5.14 (b)

Fig. 5.14 (a) and (b) TP with lumped and distributed at

y_0= [π /10; π /10; π /10; 0; 0; 0];

Figure 5.15 (a), (b) shows the Poincare map for Triple pendulum for lumped and distributed system at initial condition for π /4 degree. When comparing the fig. 5.12, it is clear that going from double pendulum to triple pendulum Poincare become more irregular.



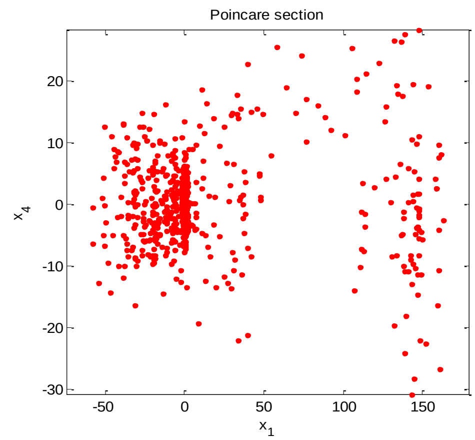
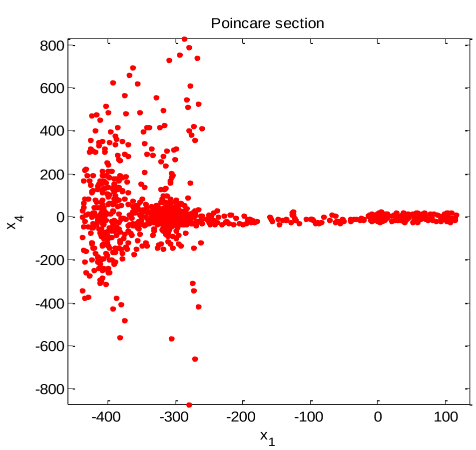


Fig. 5.15 (a)

Fig 5.15 (b)

Fig. 5.15 (a) and (b) TP with lumped and distributed at
$y_0 = [\pi /4; \pi /4; \pi /4; 0; 0; 0];$

With the further increase in angle to π /2 Poincare becomes completely irregular and that is a sign of chaotic behaviour.



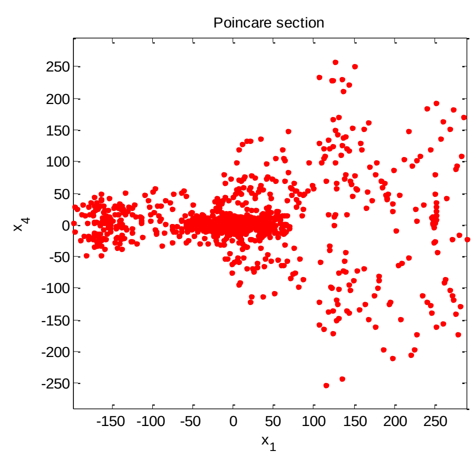


Fig. 5.16 (a)

Fig 5.16 (b)

Fig. 5.16 (a) and (b) TP with lumped and distributed at
$y_0 = [\pi /2; \pi /2; \pi /2; 0; 0; 0];$

5.5 EFFECT OF DAMPING ON THE DYNAMICS OF DP AND TP

In case of double pendulum with distributed system Rayleigh damping comes in to the picture and the effect of damping can be seen from the Eqn. (3.36).It is clear from the fig. (5.17) that magnitude will lie in the range of -0.2 to 0.2 and this is constant throughout the time when damping is zero.

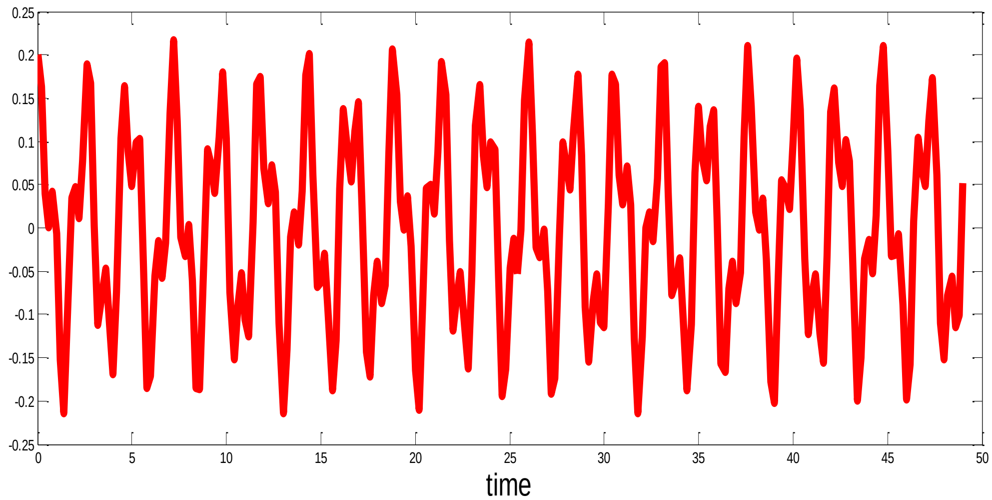
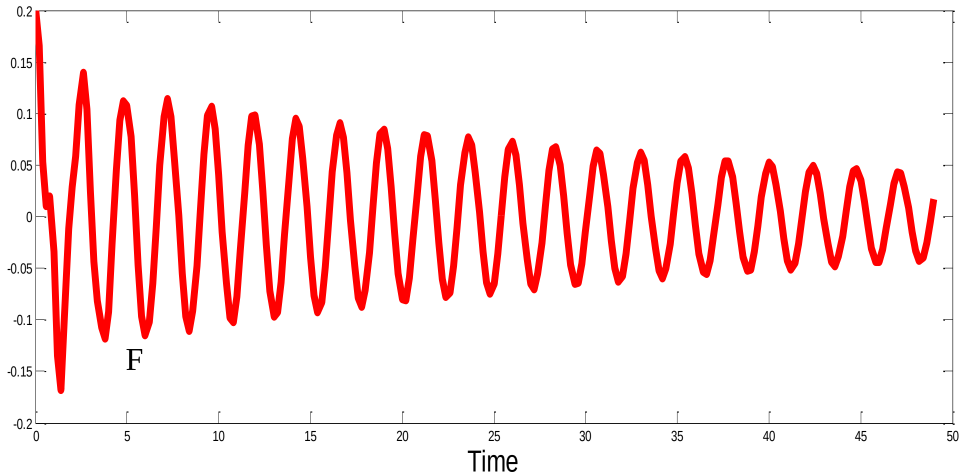


Fig 5.17 (a) Dynamics of DP at [0.2; 0.28; 0; 0] starting point without damping when mass and length are unity



5.17 (b) Dynamics of DP at starting point [0.2; 0.28; 0; 0] with damping (C1=0.1, C2=0.1) when mass and length are unity

With the introduction of small damping (C1=0.1, C2=0.1) magnitude decreases with time and it tries to make the system stable.

With the increase in damping coefficient C1=1.0, C2 =1.0, initially the magnitude is -0.1 to 0.2 but 18-20 second it tends to zero means output completely settle down.

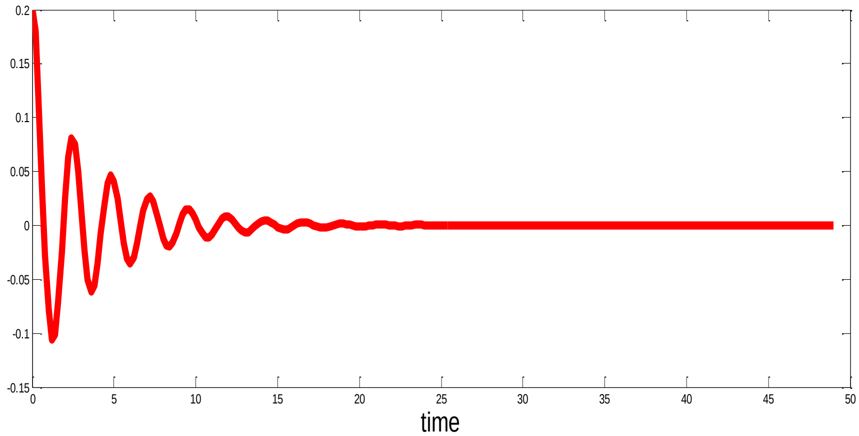


Fig 5.17 (c) Dynamics of DP at [0.2, 0.28, 0, 0] starting point with higher damping (C1=1.0, C2 =1.0) when mass and length are unity

It can be seen clearly that damping reduces the chaotic behaviour.

5.6 EFFECT OF DAMPING ON THE DYNAMICS OF TP

In case of triple pendulum with distributed system Rayleigh damping comes in to the picture and the effect of damping can be seen from the Eqn. (4.24) and Eqn. (4.25). It is clear that magnitude is around -0.5 to 0.5 throughout the time when no damping is provided.

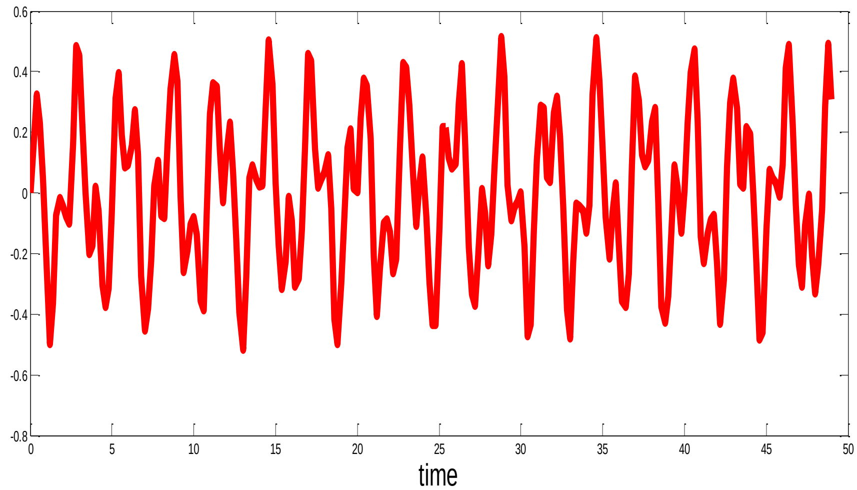


Fig. 5.18 (a) Dynamics of TP at [π /6, 0, π /6, 0, π /6, 0] starting point without damping when mass and length are unity

With the increase in damping coefficient c1=c2=c3==1, initially the magnitude is (-0.4 to 0.4) but as the time passes it decreases with time.

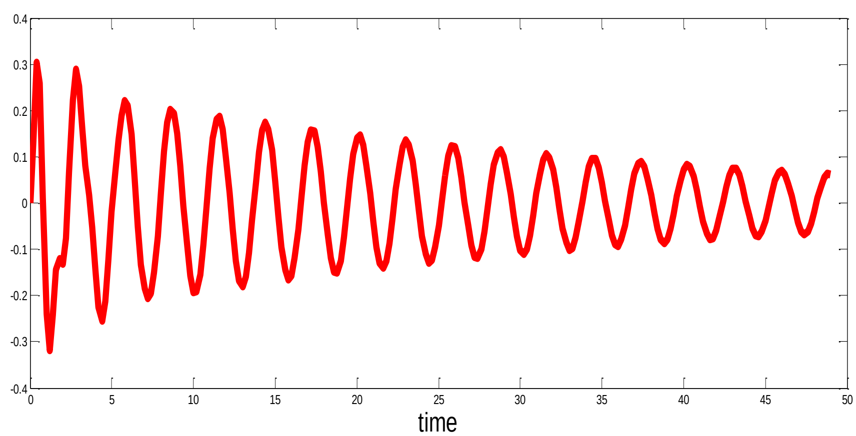


Fig. 5.18 (b) Dynamics of TP at [π /6, 0, π /6, 0, π /6, 0] starting point with damping when mass and length are unity When c1=c2=c3==1

5.7 LQR BASED CONTROL OF TRIPLE INVERED PENDULUM ON CART

The plant is linear time invariant and the state space equation of the plant is

$$\dot{X} = AX + Bu \tag{5.1}$$

It can make the performance index J achieve the minimum value

$$J = \int_0^\infty (X^T QX + u^T Ru)dt \tag{5.2}$$

Where Q is a semi definite matrix, R is a positive definite matrix, Q and R are the weighting matrixes of state variable and input variables respectively [8].For the smallest performance index function. At first Hamilton function is constructed, derivation of this was obtained and makes it equal to zero, which can determined the optimal control rate:

$$u(t) = -Kx(t) = R^{-1}B^T Px(t) \tag{5.3}$$

Where P is the only positive definite symmetric solution of which meets the Riccati Eqn. (5.4) as

$$\mathrm{PA} + \mathrm{A}^T\mathrm{P} - \mathrm{PBR}^{-1} B^T P + Q = 0 \tag{5.4}$$

In the design of the controller, one of the key problems is to select weight matrix Q and R in the quadratic performance indexes. For the Triple link inverted pendulum system, several different weighting matrices were tried and tested. The elements of the final Q matrix are larger than the elements of the R matrix. This selection translates into a controller that is more sensitive to the states of the system than the control input. The logic behind this choice is that since the main design criterion is stability, therefore the system states should dictate stability.

The elements of the Q and R matrices of the LQR selected for the system under consideration are:

Q=diag([400 6000 700 0 0 140 0 0.1]); R=1

Therefore the optimal feedback gain matrix is

$$K = 1.0e{+}003 * [-0.0200 \quad -0.3296 \quad 0.8522 \quad -3.5432 \quad -0.0458 \quad -0.1442 \quad -0.0777 \quad -0.4105]$$

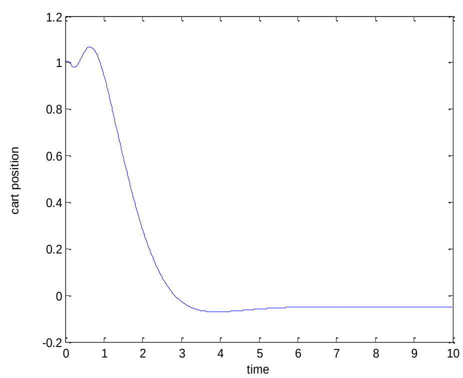
The Eigen value of system Matrix A and closed loop system matrix Ac is given below in the following table 5.3.

TABLE 5.3 (EIGEN VALUE OF A AND A_C)

S. no	Eigen value A	Eigen value Ac
1	0	-25.7123
2	9.8691	-9.6201
3	8.1493	-8.8233
4	4.3265	-8.0303
5	-10.4277	-6.1828 + 1.0041i
6	-8.6358	-6.1828 - 1.0041i
7	-6.3749	-1.2164 + 0.9812i
8	-2.2345	-1.2164 - 0.9812i

The Eigen values listed in table 5.3, it is clear that for the system matrix A, it is unstable in nature and after adding gain K, the new Eigen value of Ac are stable in nature.

Fig.5.19 (a) & 5.19 (b) shows the cart position and velocity of the system respectively. It may be seen in Fig. 5.19 (a) & 5.19 (b) that the cart stabilizes after going through initial transition. Similarly, Fig. 5.20 (a) & 5.20 (b), Fig. 5.21 (a) & 5.21 (b), and Fig. 5.22 (a) & 5.22 (b) present the plots related with angular displacement and angular velocity of the bottom link, middle link and bottom link of the TLIP. These plots also confirm that like cart, the links also stabilize with time. At the same time, it is obvious that settling time is very small. It is less than 4 seconds in almost all the cases which is a very good sign the efficacy of the LQR controller. In all simulations the systems was stable, conforming the LQR properties of the control system.



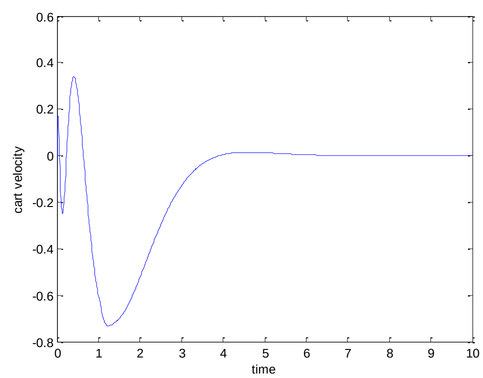
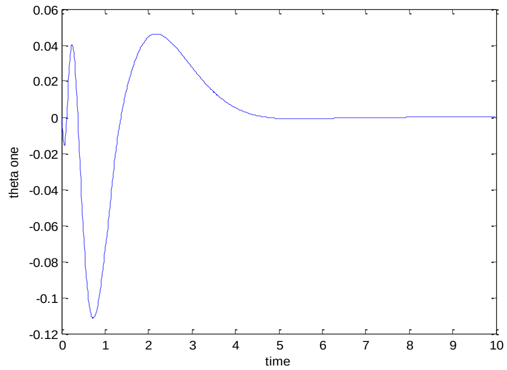


Fig 5.19 (a) Cart Position   Fig .5.19(b) Cart Velocity

Fig. 5.19 Cart Position and Cart Velocity



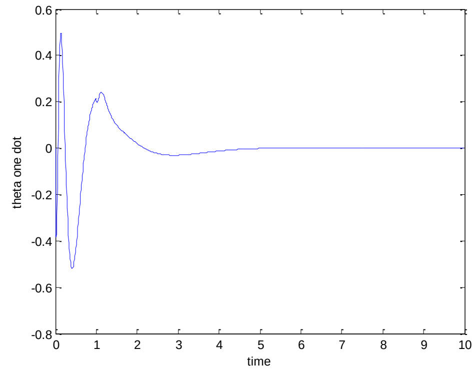
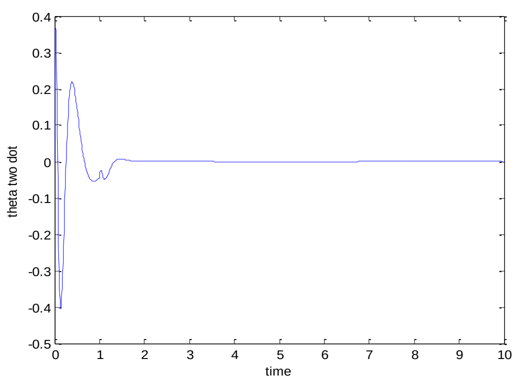


Fig. 5.20 (a) First link, angular displacement

Fig. 5.20 (b) First link, angular velocity

Fig. 5.20 First Link angular displacement and angular velocity



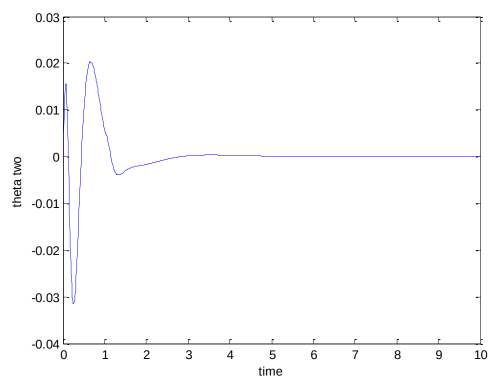
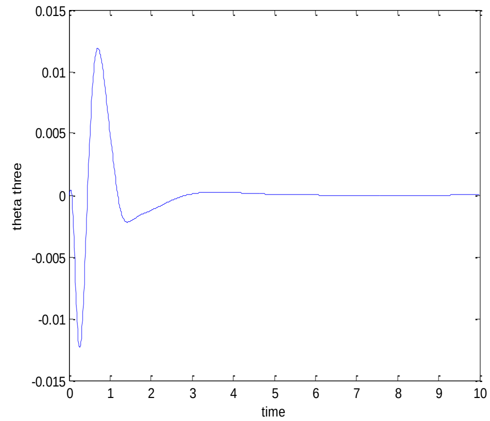


Fig 5.21 (a) Second link, angular displacement

Fig 5.23 (b) Second link, angular velocity

Fig. 5.21 Second Link angular displacement and angular velocity



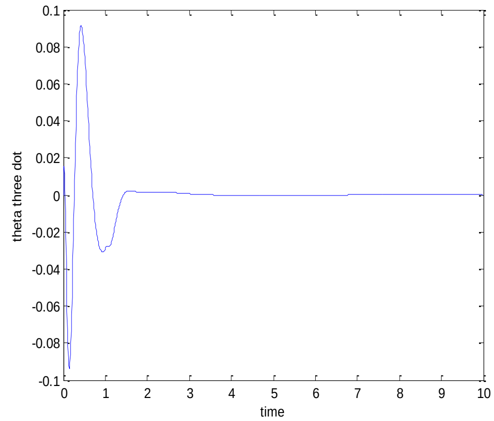


Fig 5.22 (a) Third link, angular displacement

Fig 5.22 (b) Third Link, angular velocity

Fig. 5.22 Third link angular velocity

All the parameter of the system settles down in less than 4 seconds. Simulation results clearly show the effectiveness of the proposed controller.

5.8 EXPERIMENTAL OBSERVATION OF DOUBLE AND TRIPLE PENDULUM

The setup of double and triple pendulum for distributed mass is shown in fig.(5.23, 5.24) respectively at different angle. Initially at smaller angle (around 10-15 degree) experimental setup shows the regular motion but with the slight increase in the angle (30 degree or higher) both double and triple pendulum motion goes purely randomly.

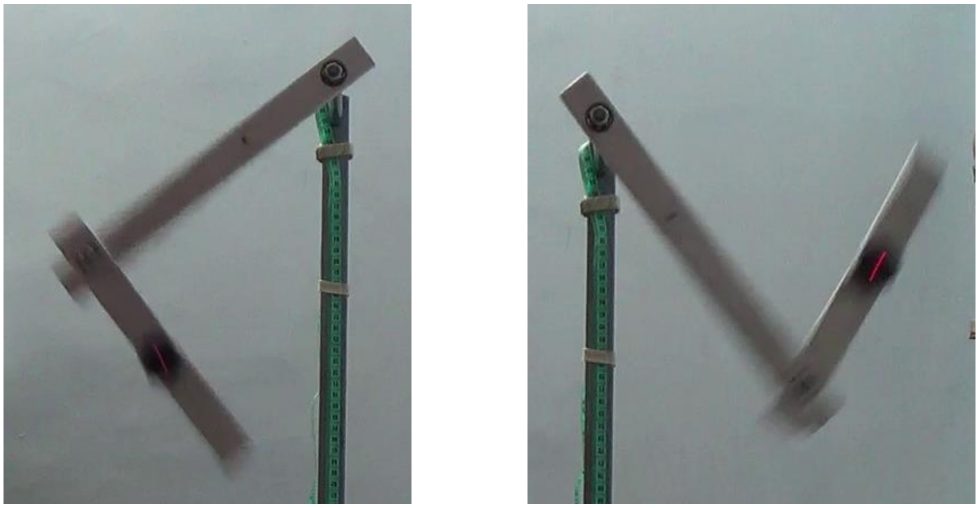


Fig 5.23 Experimental setup of double Link at different angle

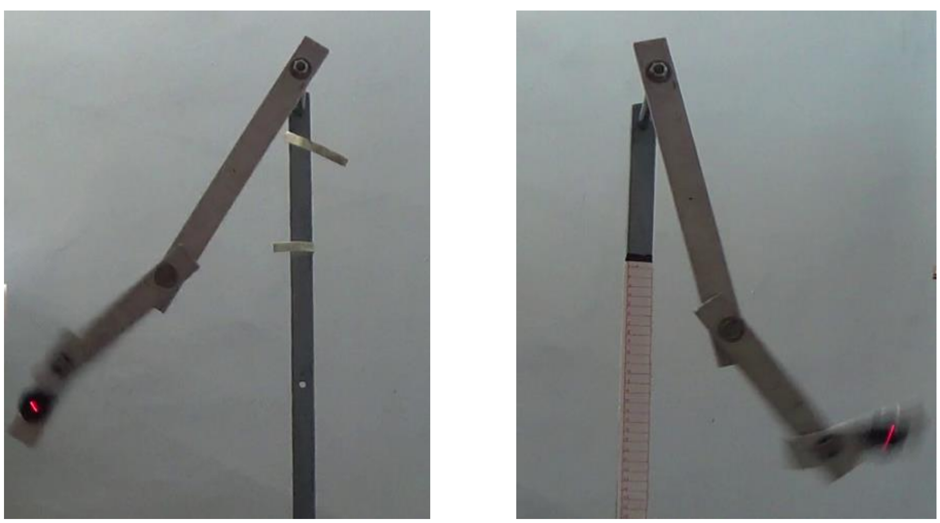


Fig 5.24 Experimental setup of triple Link at different angle

The same result has been obtained with the dynamic analysis of double and triple pendulum which validates the experimental observation.

CHAPTER 6

CONCLSUSION AND FUTURE SCOPE

6.1 CONCLUSIONS

The modelling of double and triple link pendulums for lumped and distributed system is carried out and also validated with experimental observation. The dynamic analysis has been done and following conclusions were drawn from the present study.

(1) It is observed that the bottom pendulum is most sensitive than other pendulum in both systems i.e., DP and TP.

(2) Tendency of chaos increases with degree of freedom for the same initial conditions.

(3) Poincare maps of the TP become more intricate in comparison to the DP.

(4) Damping of pivots results in regular motion of the pendulum system. In other words, introduction of damping in the multiple pendulums reduces the tendency of chaos.

(5) Moreover, there is no qualitative difference between the lumped and distributed pendulum systems as far as the dynamical behavior is concerned. Experimentally the dynamics of the double and triple pendulum has been observed.

(6) The LQR technique was used to control inverted triple pendulum. As triple inverted Pendulum is an inherently unstable system. The control technique was found to be effective.

6.2 FUTURE SCOPE

1. More experiments on DP and TP dynamical systems to gain insight into the chaotic behavior the multiple pendulums.
2. Also to design and develop the LQR based control systems so these simulation results can be verified.
3. Explore the more possibility for practical applications of the multiple pendulums.

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APPENDIX A

The coefficients for triple pendulum for distributed system are defined as follows.

$$C_{11} = I_{c1} + \frac{m_1 l_1^2}{4} + m_2 l_1^2 + m_3 l_1^2 \,, \qquad \text{A 1.1}$$

$$C_{12} = \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2) + m_3 l_1 l_2 \cos(\theta_1 - \theta_2) \,, \qquad \text{A 1.2}$$

$$C_{13} = \frac{m_3}{2} l_1 l_3 \cos(\theta_3 - \theta_1) \qquad \text{A 1.3}$$

$$C_{21} = \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2) + m_3 l_1 l_2 \cos(\theta_1 - \theta_2) \,, \qquad \text{A 1.4}$$

$$C_{22} = I_{c2} + \frac{m_2 l_2^2}{4} + m_3 l_2^2 \,, \qquad \text{A 1.5}$$

$$C_{23} = \frac{m_3}{2} l_2 l_3 \cos(\theta_2 - \theta_3) \qquad \text{A 1.6}$$

$$C_{31} = \frac{m_3}{2} l_1 l_3 \cos(\theta_3 - \theta_1) \,, \qquad \text{A 1.7}$$

$$C_{32} = \frac{m_3}{2} l_2 l_3 \cos(\theta_2 - \theta_3) \,, \qquad \text{A 1.8}$$

$$C_{33} = I_{c3} + \frac{m_3 l_3^2}{4} \qquad \text{A 1.9}$$

$$\begin{aligned} d_1 = &-\frac{m_2}{2} l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) - m_3 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) - \\ &\frac{m_3}{2} l_3 l_1 \dot{\theta}_3 \sin(\theta_3 - \theta_1)(\dot{\theta}_3 - \dot{\theta}_1) + \frac{m_2}{2} l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \\ &m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{m_3}{2} l_3 l_1 \dot{\theta}_3 \dot{\theta}_1 \sin(\theta_3 - \theta_1) + \\ &\frac{m_1 g l_1 \sin\theta_1}{2} + m_2 g l_1 \sin\theta_1 + m_3 g l_1 \sin\theta_1 + c_1 \dot{\theta}_1 \end{aligned} \qquad \text{A 1.10}$$

$$d_2 = -\frac{m_2}{2} l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) - m_3 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) -$$
$$\frac{m_3}{2} l_2 l_3 \dot{\theta}_3 \sin(\theta_2 - \theta_3)(\dot{\theta}_2 - \dot{\theta}_3) - \frac{m_2}{2} l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \qquad \text{A 1.11}$$
$$m_3 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{m_3}{2} l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) + \frac{m_2 g l_2 \sin\theta_2}{2} +$$
$$m_2 g l_2 \sin\theta_2 - c_2(\dot{\theta}_1 - \dot{\theta}_2) + c_3(\dot{\theta}_2 - \dot{\theta}_3)$$

$$d_3 = -\frac{m_3}{2}\{l_1 l_3 \dot{\theta}_1 \sin(\theta_3 - \theta_1)(\dot{\theta}_3 - \dot{\theta}_1) - l_2 l_3 \dot{\theta}_2 \sin(\theta_2 - \theta_3)(\dot{\theta}_2 - \dot{\theta}_3) +$$
$$\frac{m_3}{2}\{l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \sin(\theta_3 - \theta_1) - l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3) + \qquad \text{A 1.12}$$
$$\frac{m_3 g l_3 \sin\theta_3}{2} - c_3(\dot{\theta}_2 - \dot{\theta}_3)$$

Appendix B

The system constants for triple link inverted pendulum are defined as follows.

Constants-Value	Constants -Value
$A_1 = M + m_1 + m_2 + m_3$	$A_{19} = (m_2 l_2 + m_3 L_2) L_1$
$A_2 = m_1 l_1 + (m_2 + m_3) L_1$	$A_{20} = I_2 + m_3 L_2^{\ 2} + m_2 l_2^{\ 2}$
$A_3 = m_2 l_2 + m_3 L_2$	$A_{21} = m_3 l_3 L_2$
$A_4 = m_3 l_3$	$A_{22} = -(m_2 l_2 + m_3 L_2) L_1$
$A_5 = C_c$	$\mathrm{A}_{23} = -C_2$
$A_6 = -m_1 l_1 - (m_2 + m_3) L_1$	$A_{24} = C_2 + C_3$
$A_7 = -(m_2 l_2 + m_3 L_2)$	$A_{25} = m_3 l_3 L_2$
$A_8 = -m_3 l_3$	$A_{26} = -C_3$
$A_9 = [m_1 l_1 + (m_2 + m_3)] L_1$	$A_{27} = -g(m_2 l_2 + m_3 L_2)$
$A_{10} = I_1 + m_2 l_1^{\ 2} + (m_2 + m_3) L_1^{\ 2}$	$\mathrm{A}_{28} = m_3 l_3$
$A_{11} = (m_2 l_2 + m_3 L_2) L_1$	$A_{29} = m_3 l_3 L_1$
$A_{12} = m_3 l_3 L_1$	$A_{30} = m_3 l_3 L_2$
$A_{13} = C_1 + C_2$	$A_{31} = I_3 + m_3 l_3^{\ 2}$
$A_{14} = (m_2 l_2 + m_3 L_2) L_1$	$A_{32} = C_3$
$A_{15} = -C_2$	$A_{33} = -m_3 l_3 L_1$
$A_{16} = m_3 l_3 L_1$	$A_{34} = -g m_3 l_3$
$A_{17} = -g(m_1 l_1 + m_2 L_1 + m_3 L_1)$	$A_{35} = -m_3 l_3 L_2$
$\mathrm{A}_{18} = m_2 l_2 + m_3 L_2$	$A_{36} = -C_3$

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Ph.D PUBLICATIONS

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2. **Gupta, M. K .**, Bansal K., and Singh A.K. Mass and length dependent chaotic behavior of a double pendulum, in Advances in Control and Optimization of Dynamical Systems (ACODS), Vol. 3 Part 1 at IFAC Elsevier, D.O.I. 10.3182/20140313-3-IN-3024.00071
3. **Gupta, M. K.**, Bansal K., and Singh A.K. ,Stabilization of Triple Link Inverted Pendulum system based on LQR control Technique, Presented IEEE International Conference ICRAIE at Jaipur during May 9-11,2014
4. **Gupta, M. K.**, Gupta, Vinit , Singh, Arun K., "Linear and Non- Linear Dynamics of Double Pendula with Distributed Mass" paper selected in Sixth International Conference on Theoretical, Applied, Computational and Experimental Mechanics (ICTACEM) 29-31 Dec.2014 at IIT Kharagpur