Name: **Enrolment No:**	UPES UNIVERSITY OF TOMORROW

UNIVERSITY OF PETROLEUM AND ENERGY STUDIES
End Semester Examination, Dec 2024

Course: Linear Algebra — **Semester: I**
Program: M. Sc. Mathematics — **Time: 03 hrs.**
Course Code: MATH 7041 — **Max. Marks: 100**

Instructions: All questions are compulsory. There is an internal choice in Q9 and Q11.

SECTION A
(5Qx4M=20Marks)

S. No.		Marks	CO
Q1	Identify $n \in \mathbb{Z}_{\geq 0}$ such that $v^T v$ has n linearly independent columns where v^T is transposed vector of some nonzero $1 \times n$ vector $v \in \mathbb{R}^n$.	**4**	**CO1**
Q2	Find the nullity of matrix $f(A)$ where $f(x)$ is a real polynomial defined as $f(x) = (x-\delta)(x^3 - \alpha x^2 + \beta x - \gamma)$ and matrix $A = \begin{bmatrix} 0 & 0 & \gamma \\ 1 & 0 & -\beta \\ 0 & 1 & \alpha \end{bmatrix}$.	**4**	**CO2**
Q3	Suppose $G(n) =$ number of $n \times n$ real matrices T such that $T^2 + I_n = O_n$ (where I_n is identity and O_n is null matrix in $\mathbb{R}^n$). Determine $G(2025)$.	**4**	**CO3**
Q4	Determine the Jordan canonical form of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. Is A diagonalizable?	**4**	**CO3**
Q5	Find the value of $\lvert\lvert f(t) \rvert\rvert_1$ where $f(t) = e^{-\lvert t \rvert} \in \mathcal{C}[-1,1]$.	**4**	**CO4**

SECTION B
(4Qx10M= 40 Marks)

S. No.		Marks	CO
Q 6	Obtain the matrix representation $[T]_{\mathcal{B}}$ with respect to the ordered basis $\mathcal{B} = \{(1,0,1),(0,1,1),(1,1,0)\}$ for the linear map $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as $T(x,y,z) = (x+y, y-z, z-x)$. Is T bijective on $\mathbb{R}^3$? Justify your answer.	**10**	**CO1**
Q7	Prove that if a subspace $W \subset V$ is invariant under the linear map $T: V \longrightarrow V$ then W is invariant under $f(T)$ for any polynomial $f(t)$.	**10**	**CO2**

Q8	Suppose A is a real matrix of order $n \times n$ satisfying $A^3 = A$. Determine the smallest set S such that $trace(A) \in S$.	**10**	**CO3**
Q9	Consider the set $S = \{(1,1,-1), (1,1,1)\} \subset \mathbb{R}^3$. Find the orthogonal complement $S^\perp$ in $\mathbb{R}^3$. Also, show that $S^\perp$ is a subspace of $\mathbb{R}^3$. **OR** Prove that the vector space of continuous functions $\mathcal{C}[a,b]$ forms an inner product space with the usual inner product defined as $$< f, g > = \int_a^b f(t)g(t)dt \quad \text{where } f, g \in \mathcal{C}[a,b]$$	**10**	**CO4**
SECTION-C **(2Qx20M=40 Marks)**			
Q10	Consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix representation is $$\begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}$$ (i) Find the (generalized) eigenspaces of $\mathbb{R}^3$ (ii) Find all T-invariant subspaces W such that $W \subset \mathbb{R}^3$.	**20**	**CO3**
Q11	Consider the basis $S = \{(3,1), (2,2)\}$ in the inner product space $\mathbb{R}^2$ equipped with the conventional Euclidean inner product. Normalize the vectors of S using Gram-Schmidt orthonormalizing process. **OR** Obtain an orthonormal basis from the given basis $\left\{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ in the vector space of all 2×2 real matrices $\mathcal{M}(2, \mathbb{R})$ equipped with the inner product defined by $< A, B > = trace(B^T A)$, where B^T is the transposed matrix B.	**20**	**CO4**